

# THE BOUNDARY LAYER FORM OF THE NAVIER-STOKES EQUATIONS AND THEIR TREATMENTS



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# Outline

- Navier-Stokes equations
- Boundary layer
- Prandtl's hypothesis and the Boundary Layer Equations
- Similarity solution
- Blasius Equation
- Von Karman Integral method
- Summary

# General Equation

- For incompressible fluid:

$$\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) = -\nabla p + f + \mu \nabla^2 v$$

Where;

- $\rho$  = density of the fluid
- $f$  = body force acting on the fluid
- $p$  = static pressure
- $v$  = velocity field

# Features

- The equation is elliptic in nature
- This requires that we know the boundary conditions on the entire boundary
- Finding the analytical solution is difficult
- Approximate solutions easily solvable

# Boundary layer

- The boundary layer of a flowing fluid is the thin layer close to the wall
- In a flow field, viscous stresses are very prominent within this layer.
- Although the layer is thin, it is very important to know the details of flow within it.
- The gradient of the velocity component in a direction normal to the surface is large as compared to the gradient in the streamwise direction.

# Applications of the Boundary Layer

- Boundary layer entirely accounts for the effects of fluid viscosity
- It can be used to evaluate the shear stresses on a solid bodies and hence can be used to evaluate the total drag forces on the body in a submerged fluid
- E.g. Estimation of drag forces on a swimmer, wind tunnel testing, etc.
- We can also calculate other parameters like entrance length in pipe flow.

# Significance of the boundary layer

- Away from solid boundaries, effect of fluid viscosity negligible
- The Navier Stokes Equation reduces to Euler's Equation
- For steady case in 2 dimensions:

$$\rho(u_e \frac{\partial u_e}{\partial x} + v_e \frac{\partial v_e}{\partial y}) = -\nabla p + f$$

- Its not a good approximation near the surface because it cannot satisfy both the boundary conditions at the surface.
- Namely the no penetration and no slip boundary condition

# Boundary Conditions

- On the surface of the solid boundary:  
 $u = 0$  where  $u$  = component of velocity parallel to free-stream velocity
- In the free stream:  
 $u = U_{\infty}$  where  $U_{\infty}$  = free stream velocity
- Boundary layer thickness defined as the thickness over the solid surface over which  $u$  varies from 0 to  $0.99 U_{\infty}$



# Prandtl's hypothesis

- Ludwig Prandtl introduced the concept of boundary layer and derived the equations for boundary layer flow by correct reduction of Navier-Stokes equations.
- Prandtl said that the effect of internal friction in the fluid is significant only in a narrow region surrounding solid boundaries or bodies over which the fluid flows.
- Outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer.
- Hence flow outside the boundary layer can be taken as a potential flow.

# Governing equations

- The governing equations of motion are:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right) + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- This is the X momentum equation

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \left( \frac{\partial p}{\partial y} \right) + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

- This is the Y momentum equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- This is the continuity equation

# Scaling analysis

- We can choose characteristic scales as
- $L$  for  $x$ ,  $U_\infty$  for  $u$ ,  $\delta$  for  $y$ .
- From scaling analysis of the continuity equation we get

$$v \sim \left( U_\infty \frac{\delta}{L} \right)$$

- We see that  $\delta / L$  is an important parameter and for general case it is  $\ll 1$
- For  $\delta / L \ll 1$ , we see that  $p \sim \rho U_\infty^2$  which implies inviscid flow.
- A general conclusion is that  $\rightarrow$  axial diffusion  $\ll$  transverse diffusion

$$\frac{\partial^2(\_)}{\partial x^2} \ll \frac{\partial^2(\_)}{\partial y^2}$$

# Y momentum equation

- From the Y momentum equation we can see that  $\frac{\partial p}{\partial y} = 0 \rightarrow p$  is only a function of  $x$ .
- The pressure is imposed on the Boundary layer.
- Hence we can use the Euler's equation to get
- $$-\frac{\partial p}{\partial x} = \rho U_{\infty} \frac{\partial U_{\infty}}{\partial x}$$
- For a flat plate , it is 0 . It is non Zero for curved plates.

# X momentum equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{U_\infty^2}{L} \quad \frac{U_\infty^2}{L} \quad \frac{U_\infty^2}{L} \quad \nu \frac{U_\infty}{\delta^2}$$

- Matching the orders , we obtain that

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{\text{Re}_L}}$$

- Hence we see that the system parameter and boundary layer thickness can be compared.

# Reduced form

- Hence the reduced form of the equations are:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- With boundary conditions :  $y=0; u=0, v=0; y=\delta, u=U_\infty$
- These are non linear equations and cannot be solved directly
- The behavior can be normalized by noting that the velocity profile only gets stretched as we go along the plate
- Hence we can think of  $u/U_\infty$  to be normalized by  $\frac{y}{\delta}$   
(Where  $\delta$  is a function of  $x$ )

# Prandtl's equations' significance

- Hence the velocity can be found out using a single reduced variable which is a function of  $x$  and  $y$ . (From similarity consideration)
- It allows the no-slip boundary condition to be satisfied which constitutes a significant improvement over the potential flow analysis while solving real fluid flow problems.
- It is thus a significant improvement over the Navier Stokes equation.

# Conclusions

- At the point of separation, boundary layer thickness becomes quite large for the thin layer approximation to be valid.
- In applying the boundary layer theory  $U$  is interpreted as the fluid velocity at the wall calculated from inviscid flow considerations
- Mathematically, application of the boundary - layer theory converts the character of governing Navier-Stoke equations from elliptic to parabolic
- This allows the marching in flow direction, as the solution at any location is independent of the conditions farther downstream.



# Measure of Boundary Layer Thickness

- $u = 0.99U_\infty$  . This is however arbitrary

- **Displacement Thickness**

$$\delta^* = \int_0^{\infty} \left(1 - \frac{u}{U_\infty}\right) dy$$

- By making a correction of this distance in the inviscid flow region , we can have the same mass flux.
- This is important in design of ducts, intakes of engines, wind tunnels etc.

# Momentum Thickness

- $$\rho U_{\infty}^2 h - \int_0^h \rho u^2 dy - \rho \delta^* U_{\infty}^2 = \rho U_{\infty}^2 \theta$$

- $$\theta = \int_0^{\infty} \frac{u}{U_{\infty}} \left(1 - \frac{u}{U_{\infty}}\right) dy$$

- Hence  $\rho U_{\infty}^2 \theta$  is the momentum loss due to presence of the boundary Layer.

# Blasius Problem

- Two-dimensional, steady, incompressible flow over a flat plate at zero angle of incidence with respect to the uniform stream of velocity
- Objectives are to find out:
  - Velocity Field
  - Boundary layer thickness
  - Shear stress distribution

# Conditions

- Since it is a flow over flat plate , there is no pressure gradient term
- We can take  $\eta = y.g(x)$  &  $\eta = func(x, y)$   
$$\frac{u}{U_\infty} = f(\eta)$$
- We can thus convert the partial differential equation into an ordinary differential one by the use of a similarity parameter

# Simplification

$$\frac{\partial u}{\partial x} = U_{\infty} y f' g'$$

$$\frac{\partial u}{\partial y} = U_{\infty} f' g$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \frac{\partial u}{\partial y} = U_{\infty} f'' g^2$$

# Solving

- From the X momentum equation we get

$$v = \frac{v \frac{\partial^2 u}{\partial y^2} - u \frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

- On simplification we get:

$$v = v \frac{f''}{f'} g - U_{\infty} f y \frac{g'}{g}$$

- Taking the partial derivative of  $v$  wrt  $y$  and putting in the continuity equation;

- $$\frac{\partial v}{\partial y} = \nu g^2 \left( \frac{d\left(\frac{f'}{f''}\right)}{d\eta} \right) - U_\infty \frac{g'}{g} (f' g y + f) \quad ; \quad \frac{\partial u}{\partial x} = U_\infty y g' f'$$

- $$\frac{\frac{d(f''/f)}{d\eta}}{f} = \frac{U_\infty}{\nu} \left( \frac{g'}{g^3} \right) = k$$

- Solving  $\frac{U_\infty}{\nu} \left( \frac{\delta g'}{\delta^3} \right) = k$

- Choosing  $k=1/2$ ;

- We get  $g(x) = \sqrt{\frac{U_\infty}{\nu x}}$

- Which clearly shows that  $g(x) \sim 1/\delta$



# Blasius Equation

- Using the same value of  $k$ ,
- We get  $2F'''' + FF'' = 0$  where  $\int f d\eta = F$
- With Boundary conditions as
- $\eta = 0; F = 0, F' = 0 ; \eta = \infty, F'' = 0$
- $u = U_{\infty} f = U_{\infty} F' = \frac{\partial \psi}{\partial y} = \frac{d\psi}{d\eta} g$
- Hence  $F$  has a significance of a stream function

# Boundary conditions

- $F(0)=0 \rightarrow$  streamline at solid surface.
- $F'(0)=0 \rightarrow$  No Slip Boundary Condition.
- $F''(\infty)=0 \rightarrow$  No velocity gradient at free stream.
- This system of coupled initial value problem can be solved using RK4 method.

# Shooting method

- Initial condition  $F''(0)$  is not available with us.
- Hence we estimate a value for it and see what  $F''(\infty)$  it yields .
- According to the deviation reached, we reach at a value of  $F''(0)$
- This method is called shooting method.
- By trial and error, the value of  $F''(0)$  can be perfected.

$\eta$	$F$	$F'$	$F''$
0	0	0	0.33206
0.2	0.00664	0.006641	0.33199
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
8.8	7.07923	1.00000	0.00000

- Table showing values of  $\eta$ ,  $F$ ,  $F'$  and  $F''$

# Conclusions

- At  $u=0.99 U_{\infty}$  ;  $\eta=5$  ; hence  $\frac{\delta}{x} = 5.0 \text{Re}^{-0.5}$
- Now that the velocity profile is determined we can estimate the wall shear stress
- $\tau_w = \mu \frac{\partial u}{\partial y}_{y=0} = \mu(U_{\infty} f' g)_{\eta=0} \approx \mu U_{\infty} \times 0.3326 \times \frac{1}{\sqrt{\nu x / U_{\infty}}}$
- Where  $\eta = yg(x)$  and  $u=U_{\infty}f(\eta)$  using value of  $F''$  from the table.
- Similarly skin friction coefficient is

$$C_f = \frac{\tau_w}{1/2\rho U_{\infty}^2} = 0.664 / \text{Re}^{-0.5}$$

# Von Karman integral method

- To employ boundary layer concepts in real engineering designs, we need approximate methods that would quickly lead to an answer even if the accuracy is somewhat less.
- Von Karman and Pohlhausen devised a simplified method by satisfying only the boundary conditions of the boundary layer flow rather than satisfying Prandtl's differential equations for each and every particle within the boundary layer.
- Integrate the X momentum equation to get :

$$\int_0^{\delta} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) dy = \int_0^{\delta} \left( \frac{-1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \right) dy$$

# Some simplifications

- On simplification of the above integral and letting the limit tend to  $\infty$ , we obtain: 
$$\frac{d}{dx} \int_0^{\infty} [u(U_{\infty} - u)] dy + \frac{dU_{\infty}}{dx} \int_0^{\infty} (U_{\infty} - u) dy = \frac{\tau_w}{\rho}$$
- This equation is known as momentum integral equation for two dimensional incompressible laminar boundary layer. The same remains valid for turbulent boundary layers as well. (The only difference is in the value of wall shear stress.)
- We note that the second term is zero for a flat plate.

# Von Karman integral method

- We assume a cubic velocity profile and then carry on further integration.
- The velocity profile using suitable Boundary conditions is : 
$$\frac{u}{U_\infty} = \frac{3}{2} \left(\frac{y}{\delta}\right) - \frac{1}{2} \left(\frac{y}{\delta}\right)^3$$
- We find that  $\frac{\delta}{x} = 4.64 \text{Re}^{-0.5}$  which is a good approximation to the Blasius solution
- The Friction Factor is  $C_f = \frac{\tau}{\frac{1}{2} \rho U_\infty^2} = \frac{0.646}{\sqrt{\text{Re}_x}}$  which is also close to Blasius result.



# Summary

- The Boundary Layer considerations give us a physical insight of the flow behaviour near walls.
- Using Prandtl's hypothesis, we were able to simplify Navier Stoke's equation in the Boundary Layer.
- We converted the elliptic N.S. Equations into a Parabolic PDE.
- We solved Blasius problem using similarity solution, converting the PDE into an ODE.
- Using the Von Karman integral method we can arrive at an approximate result.
- It was found that the value of wall shear stress and Boundary Layer thickness was approximately the same for both the flows.

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- References:

NPTEL

[http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT-KANPUR/FLUID-MECHANICS/ui/Course\\_home-9.htm](http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT-KANPUR/FLUID-MECHANICS/ui/Course_home-9.htm)

**THANK YOU!!!**