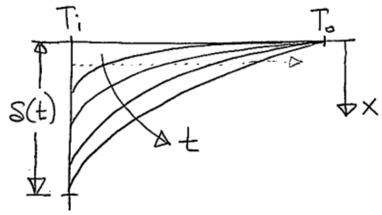
One-Dimensional Transient Conduction
(Reorganization of the Lecture Notes from Professor Nenad Miljkovic)

1-D (T=T(x,t)), $\dot{Q}''' = 0$, no convection, k=constant, α=constant

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} * \frac{\partial T}{\partial t}$$

We need 2 boundary conditions and 1 initial condition:



$$T_0 = \text{hot body temperature}$$
$$T_i = \text{initial temperature}$$

Let $\theta = \frac{T-T_0}{T_i-T_0}$

Governing PDE:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} * \frac{\partial \theta}{\partial t}$$
$$x = 0, \theta = 0$$
$$x \to \infty, \theta = 1$$
$$t = 0, \theta = 1$$

The only way to solve this is to look for a similarity variable that relates x,t. One that works and is convenient is:

$$\eta = x/f(t)$$
$$\theta = \theta(\eta)$$
$$f(t) = \text{function of time}$$

Note that similarity method is to turn PDE into ODE which we can solve.

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} * \frac{\partial \eta}{\partial x} = \frac{1}{f} * \frac{\partial \theta}{\partial \eta}$$
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{f^2} * \frac{\partial^2 \theta}{\partial \eta^2}$$
$$\frac{\partial \theta}{\partial t} = -\eta \frac{\partial \theta}{\partial \eta} * f'/f$$

Substitute back to previous PDE:

$$\frac{\partial^2 \theta}{\partial \eta^2} + \left(f * \frac{f'}{\alpha}\right) \eta * \frac{\partial \theta}{\partial \eta} = 0$$

Let $f * \frac{f'}{\alpha}$ equal to a constant.

$f * \frac{f'}{\alpha} = 2$ (can choose any arbitrary number)

$$f = 2\sqrt{\alpha t}$$
$$\eta = \frac{x}{2\sqrt{\alpha t}}$$

And

$\frac{\partial^2 \theta}{\partial \eta^2} + 2\eta * \frac{\partial \theta}{\partial \eta} = 0$ is ODE now!

Apply the boundary conditions and initial conditions:

$$\theta = \frac{T - T_0}{T_i - T_0} = \frac{2}{\sqrt{\pi}} * \int_0^{\eta} e^{-\eta^2} d\eta = \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Heat transfer during transient conduction:

$$q''_{x=0} = -k * \frac{\partial T}{\partial x}|_{x=0} = \frac{k\Delta T}{\sqrt{\pi \alpha t}}$$

There are some other cases like specified surface heat flux ($q'' = \mathrm{constant}$); convection on the surface ($q'' = h(T_\infty - T(0,t))$); energy pulse at surface (like a laser pulse with no losses, all heat goes into the solid)

① Specified surface heat flux, $q'' = \text{constant}$

$$T(x,t) - T_i = \frac{q''}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\, e^{\left(-\frac{x^2}{4\alpha t}\right)} - x\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

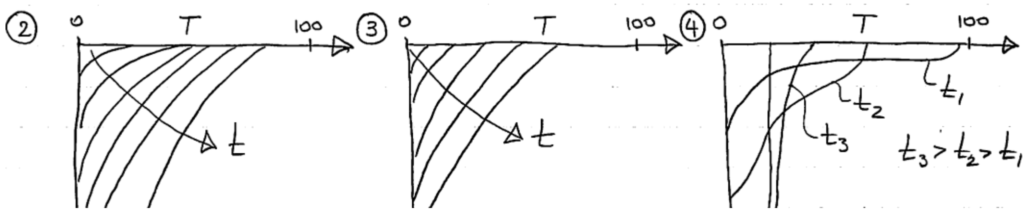
② Convection on the surface, $q'' = h(T_\infty - T(0,t))$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - e^{\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)} \cdot \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

③ Energy pulse at surface, $e_s = \text{constant}$ $\left[\frac{J}{m^2}\right]$ @ $t = 0$

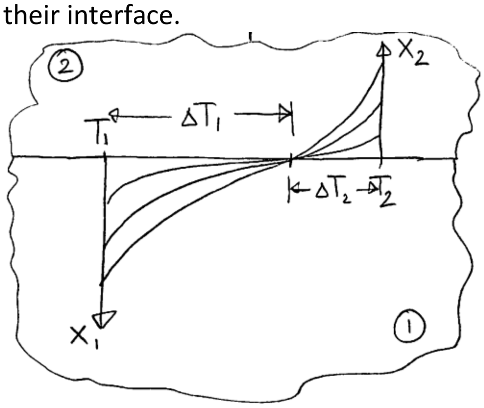
Like a laser pulse, no losses, all heat goes into the solid

$$T(x,t) - T_i = \frac{e_s}{k\sqrt{\pi t/\alpha}}\, e^{\left(-\frac{x^2}{4\alpha t}\right)}$$



Contact of two semi-infinite solids

Two bodies at T_1 and T_2 brought into contact, they instantly achieve a temperature that is constant at their interface.



$$T_{interface} = \mathrm{constant}$$

$T_{interface}$ will develop very quickly after the two bodies touch and remain the same.

$$\Delta T = T_1 - T_2$$

$$q_1'' = \frac{k_1 \Delta T_1}{\sqrt{\pi \alpha_1 t}} = q_2'' = \frac{k_2 \Delta T_2}{\sqrt{\pi \alpha_2 t}}$$

Therefore $(k_1 \rho_1 C_1)^{0.5} \Delta T_1 = (k_2 \rho_2 C_2)^{0.5} \Delta T_2$ (1)

$\Delta T_1 + \Delta T_2 = \Delta T$ (2)

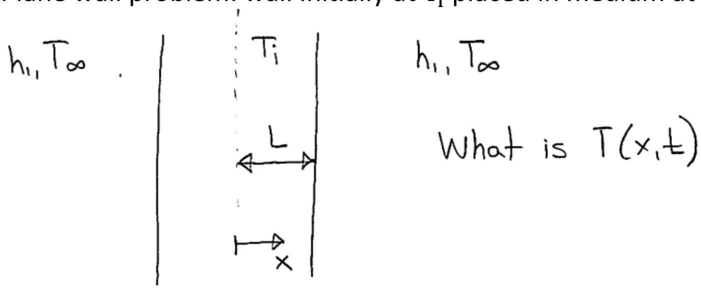
$$\Delta T_1 = \frac{\Delta T}{1 + \frac{(k\rho C)_1{}^{0.5}}{(k\rho C)_2{}^{0.5}}}$$

$$q''_{x=0} = \frac{(k\rho C)_1{}^{0.5} (k\rho C)_2{}^{0.5}}{(k\rho C)_1{}^{0.5} + (k\rho C)_2{}^{0.5}} * \frac{\Delta T}{\sqrt{\pi t}}$$

This is fundamentally why when you touch certain objects in a room, they feel "colder" than others, even though they are at the same temperature.

Transient heat conduction in finite bodies

Plane wall problem: wall initially at T_i placed in medium at T_∞ and h_1 on outside.



Heat equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} * \frac{\partial T}{\partial t}$$

$$T(x, t = 0) = T_i$$

$$\frac{\partial T}{\partial x}|_{x=0} = 0$$

$$-k * \frac{\partial T}{\partial x}|_{x=L} = h(T - T_\infty)$$

Non-dimensionalize

Let $\bar{x} = \frac{x}{L}$, $\theta = \frac{T - T_\infty}{T_i - T_\infty}$

$$\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{L^2}{\alpha} * \frac{\partial \theta}{\partial t}$$

Let $\partial \tau = \frac{\alpha}{L^2} * \partial t$

$$\tau = F_o = \frac{\alpha t}{L^2} = \frac{\text{diffusive heat conduction rate}}{\text{heat storage rate (transient)}} = \text{dimentionless time in heat transfer}$$

F_o is Fourier number

Non-dimentionalize the second boundary condition:

$$\frac{\partial \theta(1, \tau)}{\partial \bar{x}} = -\frac{hL}{k} * \theta(1, \tau)$$

Note that $\frac{hL}{k} = Bi_L$ => Biot number=$\frac{\text{conduction resistance}}{\text{convection resistance}}$

Other B.C. & I.C. are:

$$\frac{\partial \theta(0, \tau)}{\partial \bar{x}} = 0$$

$$\theta(\bar{x}, 0) = 1$$

To solve the equations, we need to use separation of variables:

$$\theta(\bar{x}, \tau) = F(\bar{x}) * G(\tau)$$

Solution:

$$\theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n \bar{x})$$

$$A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)}$$

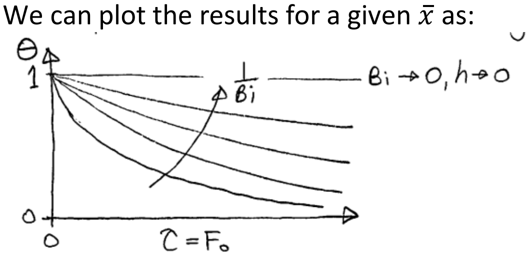
$\lambda_n \tan(\lambda_n) = Bi$ (eigenfunction with eigenvalues λ_n)

Since the solution involves an infinite series, it is not very useful to solve analytically. However, good approximation is made with the first few terms since the rest decay rapidly due to the $e^{-\lambda_n^2 \tau}$ term.

Tabular solution:

We know the solution is: $\theta = f(\bar{x}, Bi, F_o)$

We can plot the results for a given $\bar{x}$ as:



$Bi_L = \frac{hL}{k}$

Similarly if we have a cylinder.