ORDINARY DIFFERENTIAL EQUATIONS

GABRIEL NAGY

Mathematics Department,
Michigan State University,
East Lansing, MI, 48824.

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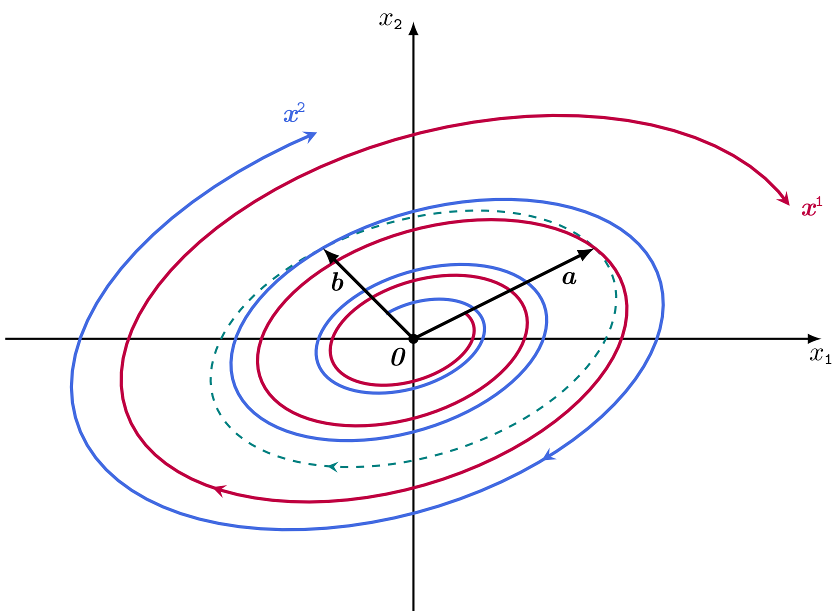
SUMMARY. This is an introduction to ordinary differential equations. We describe the main ideas to solve certain differential equations, like first order scalar equations, second order linear equations, and systems of linear equations. We use power series methods to solve variable coefficients second order linear equations. We introduce Laplace transform methods to find solutions to constant coefficients equations with generalized source functions. We provide a brief introduction to boundary value problems, Sturm-Liouville problems, and Fourier Series expansions. We end these notes solving our first partial differential equation, the Heat Equation. We use the method of separation of variables, hence solutions to the partial differential equation are obtained solving infinitely many ordinary differential equations.



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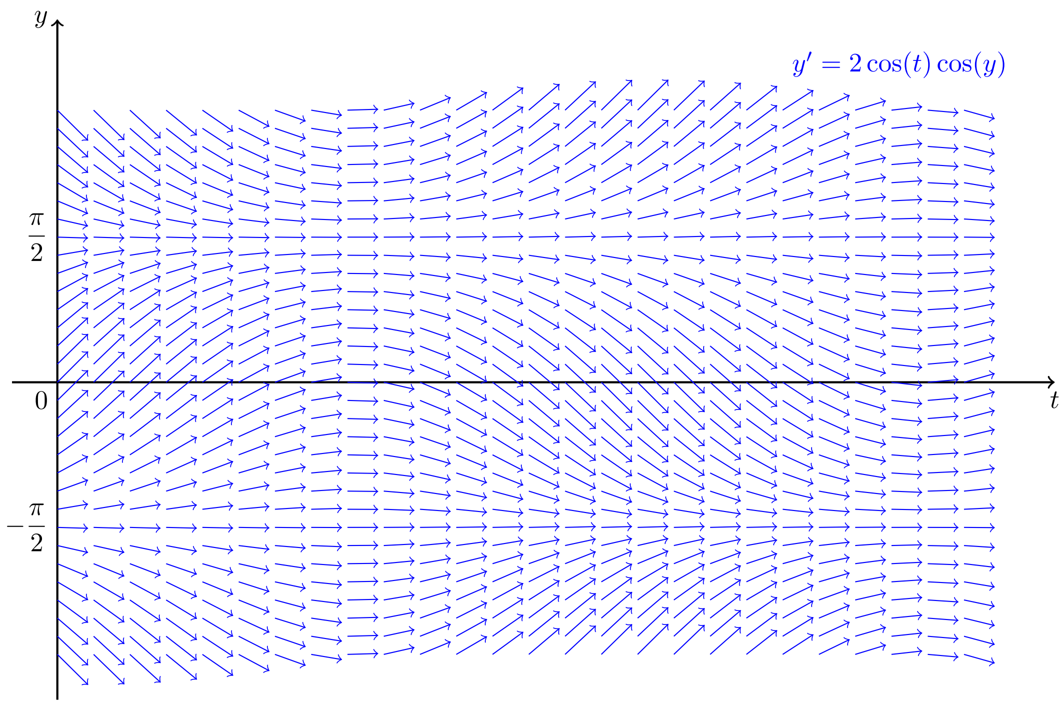
Chapter 1. First Order Equations

We start our study of differential equations in the same way the pioneers in this field did. We show particular techniques to solve particular types of first order differential equations. The techniques were developed in the eighteen and nineteen centuries and the equations include linear equations, separable equations, Euler homogeneous equations, and exact equations. Soon this way of studying differential equations reached a dead end. Most of the differential equations cannot be solved by any of the techniques presented in the first sections of this chapter. People then tried something different. Instead of solving the equations they tried to show whether an equation has solutions or not, and what properties such solution may have. This is less information than obtaining the solution, but it is still valuable information. The results of these efforts are shown in the last sections of this chapter. We present Theorems describing the existence and uniqueness of solutions to a wide class of differential equations.



1.1. Linear Constant Coefficients Equations

1.1.1. **Overview of Differential Equations.** A differential equation is an equation, the unknown is a function, and both the function and its derivatives may appear in the equation. Differential equations are essential for a mathematical description of nature. They are at the core of many physical theories: Newton's and Lagrange equations for classical mechanics, Maxwell's equations for classical electromagnetism, Schrödinger's equation for quantum mechanics, and Einstein's equation for the general theory of gravitation, to mention a few of them. The following examples show how differential equations look like.

(a) Newton's second law of motion for a single particle. The unknown is the position in space of the particle, $\boldsymbol{x}(t)$, at the time t. From a mathematical point of view the unknown is a single variable vector-valued function in space. This function is usually written as $\boldsymbol{x}$ or $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^3$, where the function domain is every $t \in \mathbb{R}$, and the function range is any point in space $\boldsymbol{x}(t) \in \mathbb{R}^3$. The differential equation is

$$m \frac{d^2\boldsymbol{x}}{dt^2}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)),$$

where the positive constant m is the mass of the particle and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is the force acting on the particle, which depends on the time t and the position in space $\boldsymbol{x}$. This is the well-known law of motion *mass times acceleration equals force.*

(b) The time decay of a radioactive substance. The unknown is a scalar-valued function $u : \mathbb{R} \to \mathbb{R}$, where $u(t)$ is the concentration of the radioactive substance at the time t. The differential equation is

$$\frac{du}{dt}(t) = -k\, u(t),$$

where k is a positive constant. The equation says the higher the material concentration the faster it decays.

(c) The wave equation, which describes waves propagating in a media. An example is sound, where pressure waves propagate in the air. The unknown is a scalar-valued function of two variables $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, where $u(t, \boldsymbol{x})$ is a perturbation in the air density at the time t and point $\boldsymbol{x} = (x, y, z)$ in space. (We used the same notation for vectors and points, although they are different type of objects.) The equation is

$$\partial_{tt} u(t, \boldsymbol{x}) = v^2 \big[\partial_{xx} u(t, \boldsymbol{x}) + \partial_{yy} u(t, \boldsymbol{x}) + \partial_{zz} u(t, \boldsymbol{x})\big],$$

where v is a positive constant describing the wave speed, and we have used the notation ∂ to mean partial derivative.

(d) The heat conduction equation, which describes the variation of temperature in a solid material. The unknown is a scalar-valued function $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, where $u(t, \boldsymbol{x})$ is the temperature at time t and the point $\boldsymbol{x} = (x, y, z)$ in the solid. The equation is

$$\partial_t u(t, \boldsymbol{x}) = k \big[\partial_{xx} u(t, \boldsymbol{x}) + \partial_{yy} u(t, \boldsymbol{x}) + \partial_{zz} u(t, \boldsymbol{x})\big],$$

where k is a positive constant representing thermal properties of the material.

The equations in examples (a) and (b) are called ***ordinary differential equations*** (ODE), since the unknown function depends on a single independent variable, t in these examples. The equations in examples (c) and (d) are called ***partial differential equations*** (PDE), since the unknown function depends on two or more independent variables, t, x, y, and z in these examples, and their partial derivatives appear in the equations.

The ***order*** of a differential equation is the highest derivative order that appears in the equation. Newton's equation in Example (a) is second order, the time decay equation in Example (b) is first order, the wave equation in Example (c) is second order is time and space variables, and the heat equation in Example (d) is first order in time and second order in space variables.

1.1.2. **Linear Equations.** A good start is a precise definition of the differential equations we are about to study in this Chapter. We use primes to denote derivatives,

$$\frac{dy}{dt}(t) = y'(t).$$

This is a compact notation and we use it when there is no risk of confusion.

Definition 1.1.1. *A **first order ordinary differential equation** in the unknown y is*

$$y'(t) = f(t, y(t)), \tag{1.1.1}$$

*where $y : \mathbb{R} \to \mathbb{R}$ is the unknown function and $f : \mathbb{R}^2 \to \mathbb{R}$ is a given function. The equation in (1.1.1) is called **linear** iff the function with values $f(t, y)$ is linear on its second argument; that is, there exist functions $a, b : \mathbb{R} \to \mathbb{R}$ such that*

$$y'(t) = a(t)\, y(t) + b(t), \qquad f(t, y) = a(t)\, y + b(t). \tag{1.1.2}$$

A different sign convention for Eq. (1.1.2) may be found in the literature. For example, Boyce-DiPrima [3] writes it as $y' = -a\, y + b$. The sign choice in front of function a is just a convention. Some people like the negative sign, because later on, when they write the equation as $y' + a\, y = b$, they get a plus sign on the left-hand side. In any case, we stick here to the convention $y' = ay + b$.

A linear first order equation has ***constant coefficients*** iff both functions a and b in Eq. (1.1.2) are constants. Otherwise, the equation has ***variable coefficients***.

EXAMPLE 1.1.1:

(a) An example of a first order linear ODE is the equation

$$y'(t) = 2\, y(t) + 3.$$

In this case, the right-hand side is given by the function $f(t, y) = 2y + 3$, where we can see that $a(t) = 2$ and $b(t) = 3$. Since these coefficients do not depend on t, this is a constant coefficients equation.

(b) Another example of a first order linear ODE is the equation

$$y'(t) = -\frac{2}{t}\, y(t) + 4t.$$

In this case, the right hand side is given by the function $f(t, y) = -2y/t + 4t$, where $a(t) = -2/t$ and $b(t) = 4t$. Since the coefficients are nonconstant functions of t, this is a variable coefficients equation. $\lhd$

A function $y : D \subset \mathbb{R} \to \mathbb{R}$ is ***solution*** of the differential equation in (1.1.1) iff the equation is satisfied for all values of the independent variable t in the domain D of the function y.

EXAMPLE 1.1.2: Show that $y(t) = e^{2t} - \dfrac{3}{2}$ is solution of the equation $y'(t) = 2\, y(t) + 3$.

SOLUTION: We need to compute the left and right-hand sides of the equation and verify they agree. On the one hand we compute $y'(t) = 2e^{2t}$. On the other hand we compute

$$2\, y(t) + 3 = 2\Big(e^{2t} - \frac{3}{2}\Big) + 3 = 2e^{2t}.$$

We conclude that $y'(t) = 2\, y(t) + 3$ for all $t \in \mathbb{R}$. $\lhd$

1.1.3. **Linear Equations with Constant Coefficients.** Constant coefficient equations are simpler to solve than variable coefficient ones. There are many ways to solve them. Integrating each side of the equation, however, *does not* work. For example, take the equation

$$y' = 2\,y + 3,$$

and integrate on both sides,

$$\int y'(t)\,dt = 2\int y(t)\,dt + 3t + c, \quad c \in \mathbb{R}.$$

The Fundamental Theorem of Calculus implies $y(t) = \int y'(t)\,dt$. Using this equality in the equation above we get

$$y(t) = 2\int y(t)\,dt + 3t + c.$$

We conclude that integrating both sides of the differential equation is not enough to find the solution y. We still need to find a primitive of y. Since we do not know y, we cannot find its primitive. The only thing we have done here is to rewrite the original differential equation as an integral equation. That is why integrating both side of a linear equation does not work.

One needs a better idea to solve a linear differential equation. We describe here one possibility, the *integrating factor method*. Multiply the differential equation

$$y' = ay + b$$

by a particular function, called the integrating factor. Choose the integrating factor having one important property. The differential equation is transformed as follows,

$$y' = ay + b \quad \to \quad \frac{d\psi}{dt}(t, y(t)) = 0,$$

that is, as total derivative of a function ψ. This function depends on t and y and is called a *potential function*. Integrating the differential equation is now trivial, the potential function must be a constant, $\psi(t, y(t)) = c$. A solution y of the differential equation is given implicitly by the equation

$$\psi(t, y(t)) = c.$$

This whole idea is called the integrating factor method.

In the next Section we generalize this idea to find solutions linear equations with variable coefficients. In Section 1.4 we generalize this idea to certain nonlinear differential equations. We now state in a theorem a precise formula for the solutions of constant coefficient linear equations.

Theorem 1.1.2 (Constant Coefficients). *The linear differential equation*

$$y'(t) = a\,y(t) + b \tag{1.1.3}$$

where $a \neq 0$, b are constants, has infinitely many solutions labeled by $c \in \mathbb{R}$ as follows,

$$y(t) = c\,e^{at} - \frac{b}{a}. \tag{1.1.4}$$

Remark: Eq. (1.1.4) is called the ***general solution*** of the differential equation in (1.1.3). Theorem 1.1.2 says that Eq. (1.1.3) has infinitely many solutions, one solution for each value of the constant c, which is not determined by the equation. This is reasonable. Since the differential equation contains one derivative of the unknown function y, finding a solution of the differential equation requires to compute an integral. Every indefinite integration introduces an integration constant. This is the origin of the constant c above.

Proof of Theorem 1.1.2: The *integrating factor method* is the key idea in the proof of Theorem 1.1.2. Write the differential equation with the unknown function on one side only,

$$y'(t) - a\,y(t) = b,$$

and then multiply the differential equation by the exponential e^{-at}, where the exponent is negative the constant coefficient a in the differential equation multiplied by t. This exponential function is an ***integrating factor*** for the differential equation. The reason to choose this particular function, e^{-at}, is explained in Lemma 1.1.3, below. The result is

$$\big[y'(t) - a\,y(t)\big]\,e^{-at} = b\,e^{-at} \quad \Leftrightarrow \quad e^{-at}\,y'(t) - a\,e^{-at}\,y(t) = b\,e^{-at}.$$

This exponential is chosen because of the following property,

$$-a\,e^{-at} = \big(e^{-ay}\big)'.$$

Introducing this property into the differential equation we get

$$e^{-at}\,y'(t) + \big(e^{-at}\big)'\,y(t) = b\,e^{-at}.$$

Now the product rule for derivatives implies that the left-hand side above is a total derivative,

$$\big[e^{-at}\,y(t)\big]' = b\,e^{-at}.$$

At this point we can go one step further, writing the right-hand side in the differential equation as $b\,e^{-at} = \Big[-\frac{b}{a}\,e^{-at}\Big]'$. We obtain

$$\Big[e^{-at}\,y(t) + \frac{b}{a}\,e^{-at}\Big]' = 0 \quad \Leftrightarrow \quad \Big[\Big(y(t) + \frac{b}{a}\Big)\,e^{-at}\Big]' = 0.$$

Hence, the *whole differential equation is a total derivative*. The whole differential equation is the total derivative of the function,

$$\psi(t, y(t)) = \Big(y(t) + \frac{b}{a}\Big)\,e^{-at},$$

called ***potential function***. The equation now has the form

$$\frac{d\psi}{dt}(t, y(t)) = 0.$$

It is simple to integrate the differential equation when written using a potential function,

$$\psi(t, y(t)) = c \quad \Leftrightarrow \quad \Big(y(t) + \frac{b}{a}\Big)\,e^{-at} = c.$$

Simple algebraic manipulations imply that

$$y(t) = c\,e^{at} - \frac{b}{a}.$$

This establishes the Theorem. □

Remarks:

(a) Why do we start the Proof of Theorem 1.1.2 multiplying the equation by the function e^{-at}? At first sight it is not clear where this idea comes from. In Lemma 1.1.3 we show that functions proportional to e^{-at} are the only functions having the property needed to be an integrating factor. In Lemma 1.1.3 we multiply the differential equation by a function μ and we find that this function must be $\mu(t) = c\,e^{-at}$.

(b) Since the function μ is used to multiply the original differential equation, we can freely choose any normalization for μ, such as $\mu(0) = 1$. Any other integrating factor differs from this μ by a multiplicative constant.

(c) It is important we understand the origin of the integrating factor e^{-at} in order to extend results from constant coefficients equations to variable coefficients equations.

The following Lemma states that those functions proportional to e^{-at} are integrating factors for the differential equation in (1.1.3).

Lemma 1.1.3 (Integrating Factor). *Given any differentiable function y and constant a, every function μ satisfying*

$$(y' - ay)\mu = (y\mu)',$$

must be given by the expression below, for any $c \in \mathbb{R}$,

$$\mu(t) = c\, e^{-at}.$$

Proof of Lemma 1.1.3: Multiply Eq. (1.1.3) by a nonvanishing but otherwise arbitrary function with values $\mu(t)$, and order the terms in the equation as follows

$$\mu(y' - a\,y) = b\,\mu. \tag{1.1.5}$$

The key idea of the proof is to choose the function μ such that the following equation holds

$$\mu(y' - a\,y) = \big(\mu y\big)'. \tag{1.1.6}$$

This Eq. (1.1.6) is an equation for μ. To see that this is the case, rewrite it as follows,

$$\mu\, y' - a\,\mu y = \mu'\, y + \mu\, y' \quad\Leftrightarrow\quad -a\,\mu y = \mu'\, y \quad\Leftrightarrow\quad -a\,\mu = \mu'.$$

The function y does not appear on the equation on the far right above, making it an equation for μ. So the same is true for Eq. (1.1.6). The equation above can be solved for μ as follows:

$$\frac{\mu'}{\mu} = -a \quad\Leftrightarrow\quad \big[\ln(\mu)\big]' = -a.$$

Integrate the equation above,

$$\ln(\mu) = -at + c_0 \quad\Leftrightarrow\quad e^{\ln(\mu)} = e^{-at+c_0} = e^{-at}e^{c_0},$$

where c_0 is an arbitrary constant. Denoting $c = e^{c_0}$, the integrating factor is the nonvanishing function

$$\mu(t) = c\, e^{-at}.$$

This establishes the Lemma. □

We first solve the problem in Example 1.1.3 below using the formula in Theorem 1.1.2.

EXAMPLE 1.1.3: Find all solutions to the constant coefficient equation

$$y' = 2y + 3 \tag{1.1.7}$$

SOLUTION: The equation above is the particular case of Eq. (1.1.3) given by $a = 2$ and $b = 3$. Therefore, using these values in the expression for the solution given in Eq. (1.1.4) we obtain

$$y(t) = ce^{2t} - \frac{3}{2}.$$

◁

We now solve the same problem above following the steps given in the proof of Theorem 1.1.2. In this way we see how the ideas in the proof of the Theorem work in a particular example.

EXAMPLE 1.1.4: Find all solutions to the constant coefficient equation

$$y' = 2y + 3 \tag{1.1.8}$$

Solution: Write down the equation in (1.1.8) as follows,

$$y' - 2y = 3.$$

Multiply this equation by the exponential e^{-2t}, that is,

$$e^{-2t}y' - 2\,e^{-2t}\,y = 3\,e^{-2t} \quad\Leftrightarrow\quad e^{-2t}y' + \big(e^{-2t}\big)'\,y = 3\,e^{-2t}.$$

The equation on the far right above is

$$(e^{-2t}\,y)' = 3\,e^{-2t}.$$

Rewrite the right hand side above,

$$(e^{-2t}\,y)' = \Big(-\frac{3}{2}\,e^{-2t}\Big)'.$$

Moving terms and reordering factors we get

$$\Big[\Big(y + \frac{3}{2}\Big)\,e^{-2t}\Big]' = 0.$$

Now the equation is easy to integrate,

$$\Big(y + \frac{3}{2}\Big)\,e^{-2t} = c.$$

So we get the solutions

$$y(t) = c\,e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}. \qquad \triangleleft$$

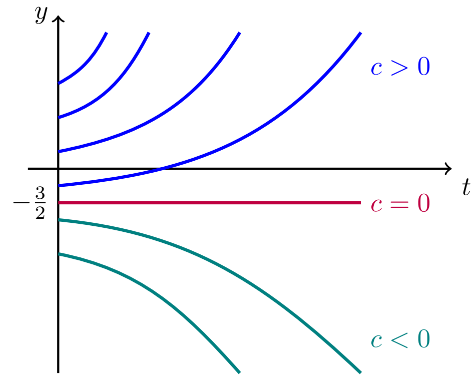


Figure 1. A few solutions to Eq. (1.1.8) for different c.

1.1.4. **The Initial Value Problem.** Sometimes in physics one is not interested in all solutions to a differential equation, but only in those solutions satisfying an extra condition. For example, in the case of Newton's second law of motion for a point particle, one could be interested only in those solutions satisfying an extra condition: At an initial time the particle must be at a specified initial position. Such condition is called an initial condition, and it selects a subset of solutions of the differential equation. An initial value problem means to find a solution to both a differential equation and an initial condition.

Definition 1.1.4. *The* ***initial value problem*** *(IVP) for a constant coefficients first order linear ODE is the following: Given a, b, t_0, $y_0 \in \mathbb{R}$, find a solution $y : \mathbb{R} \to \mathbb{R}$ of the problem*

$$y' = a\,y + b, \qquad y(t_0) = y_0. \tag{1.1.9}$$

The second equation in (1.1.9) is called the *initial condition* of the problem. Although the differential equation in (1.1.9) has infinitely many solutions, the associated initial value problem has a unique solution.

Theorem 1.1.5 (Constant Coefficients IVP). *The initial value problem in (1.1.9), for given constants $a, b, t_0, y_0 \in \mathbb{R}$, and $a \neq 0$, has the unique solution*

$$y(t) = \Big(y_0 + \frac{b}{a}\Big)e^{a(t-t_0)} - \frac{b}{a}. \tag{1.1.10}$$

It is common the case $t_0 = 0$. The initial condition is $y(0) = y_0$ and the solution y is

$$y(t) = \Big(y_0 + \frac{b}{a}\Big)e^{at} - \frac{b}{a}.$$

To prove Theorem 1.1.5 we use Theorem 1.1.2 to write the general solution of the differential equation. Then the initial condition fixes the integration constant c.

Proof of Theorem 1.1.5: The general solution of the differential equation in (1.1.9) is given in Eq. (1.1.4) for any choice of the integration constant c,

$$y(t) = c\,e^{at} - \frac{b}{a}.$$

The initial condition determines the value of the constant c, as follows

$$y_0 = y(t_0) = c\,e^{at_0} - \frac{b}{a} \quad \Leftrightarrow \quad c = \Big(y_0 + \frac{b}{a}\Big)e^{-at_0}.$$

Introduce this expression for the constant c into the differential equation in Eq. (1.1.9),

$$y(t) = \Big(y_0 + \frac{b}{a}\Big)e^{a(t-t_0)} - \frac{b}{a}.$$

This establishes the Theorem. □

EXAMPLE 1.1.5: Find the unique solution of the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1. \tag{1.1.11}$$

SOLUTION: Every solution of the differential equation is given by $y(t) = ce^{2t} - (3/2)$, where c is an arbitrary constant. The initial condition in Eq. (1.1.11) determines the value of c,

$$1 = y(0) = c - \frac{3}{2} \quad \Leftrightarrow \quad c = \frac{5}{2}.$$

Then, the unique solution to the IVP above is $y(t) = \dfrac{5}{2}\,e^{2t} - \dfrac{3}{2}$. ◁

EXAMPLE 1.1.6: Find the solution y to the initial value problem

$$y' = -3y + 1, \qquad y(0) = 1.$$

SOLUTION: Write the differential equation as $y' + 3\,y = 1$. Multiplying the equation by the exponential e^{3t} converts the left-hand side above into a total derivative,

$$e^{3t}y' + 3\,e^{3t}\,y = e^{3t} \quad \Leftrightarrow \quad e^{3t}y' + \big(e^{3t}\big)'\,y = e^{3t}.$$

This is the key idea, because the derivative of a product implies

$$\big[e^{3t}\,y\big]' = e^{3t}.$$

The exponential e^{3t} is an integrating factor. Integrate on both sides of the equation,

$$e^{3t}\,y = \frac{1}{3}e^{3t} + c.$$

So every solution of the differential equation above is given by

$$y(t) = c\,e^{-3t} + \frac{1}{3}, \qquad c \in \mathbb{R}.$$

The initial condition $y(0) = 2$ selects only one solution:

$$1 = y(0) = c + \frac{1}{3} \quad \Rightarrow \quad c = \frac{2}{3}.$$

We conclude that $y(t) = \dfrac{2}{3}\,e^{-3t} + \dfrac{1}{3}$. ◁

Notes.

This section corresponds to Boyce-DiPrima [3] Section 2.1, where both constant and variable coefficient equations are studied. Zill and Wright give a more concise exposition in [17] Section 2.3, and a one page description is given by Simmons in [10] in Section 2.10.

1.1.5. **Exercises.**

1.1.1.- Verify that $y(t) = (t+2)\,e^{2t}$ is solution of the IVP

$$y' = 2y + e^{2t}, \quad y(0) = 2.$$

1.1.2.- Follow the steps below to find all solutions of

$$y' = -4y + 2$$

(a) Find the integrating factor μ.
(b) Write the equations as a total derivative of a function ψ, that is

$$y' = -4y + 2 \quad \Leftrightarrow \quad \psi' = 0.$$

(c) Integrate the equation for ψ.
(d) Compute y using part (c).

1.1.3.- Find all solutions of

$$y' = 2y + 5$$

1.1.4.- Find the solution of the IVP

$$y' = -4y + 2, \quad y(0) = 5.$$

1.1.5.- Find the solution of the IVP

$$\frac{dy}{dt}(t) = 3\,y(t) - 2, \quad y(1) = 1.$$

1.1.6.- Express the differential equation

$$y' = 6\,y + 1 \tag{1.1.12}$$

as a total derivative of a potential function $\psi(t,y)$, that is, find ψ satisfying

$$y' = 6\,y + 1 \quad \Leftrightarrow \quad \psi' = 0.$$

Integrate the equation for the potential function ψ to find all solutions y of Eq. (1.1.12).

1.1.7.- Find the solution of the IVP

$$y' = 6\,y + 1, \quad y(0) = 1.$$

1.2. Linear Variable Coefficients Equations

We presented first order, linear, differential equations in Section 1.1. When the equation has constant coefficients we found an explicit formula for all solutions, Eq. (1.1.3) in Theorem 1.1.2. We learned that an initial value problem for these equations has a unique solution, Theorem 1.1.5. Here we generalize these results to variable coefficients equations,

$$y'(t) = a(t)\, y(t) + b(t),$$

where a, $b : (t_1, t_2) \to \mathbb{R}$ are continuous functions. We do it generalizing the integrating factor method from constant coefficients to variable coefficients equations. We will end this Section introducing the Bernoulli equation, which is a nonlinear differential equation. This nonlinear equation has a particular property: It can be transformed into a linear equation by and appropriate change in the unknown function. One then solves the linear equation for the changed function using the integrating factor method. Finally one transforms back the changed function into the original function.

1.2.1. Linear Equations with Variable Coefficients.

We start this section generalizing Theorem 1.1.2 from constant coefficients equations to variable coefficients equations.

Theorem 1.2.1 (Variable coefficients). *If the functions a, b are continuous, then*

$$y' = a(t)\, y + b(t), \tag{1.2.1}$$

has infinitely many solutions and every solution, y, can be labeled by $c \in \mathbb{R}$ as follows

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt, \tag{1.2.2}$$

where we introduced the function $A(t) = \int a(t)\, dt$, any primitive of the function a.

Remark: In the particular case of constant coefficients we see that a primitive for the constant function $a \in \mathbb{R}$ is $A(t) = at$, while

$$e^{A(t)} \int e^{-A(t)}\, b(t)\, dt = e^{at} \int b\, e^{-at}\, dt = e^{at}\Big(-\frac{b}{a}\, e^{-at}\Big) = -\frac{b}{a},$$

hence we recover the expression $y(t) = c\, e^{at} - \dfrac{b}{a}$ given in Eq. (1.1.3).

Proof of Theorem 1.2.1: We generalize the integrating factor method from constant coefficients to variable coefficients equations. Write down the differential equation as

$$y' - a(t)\, y = b(t).$$

Let $A(t) = \int a(t)\, dt$ be any primitive (also called antiderivative) of function a. Multiply the equation above by the function $e^{-A(t)}$, called the integrating factor,

$$\big[y' - a(t)\, y\big]\, e^{-A(t)} = b(t)\, e^{-A(t)} \quad \Leftrightarrow \quad e^{-A(t)} y' - a(t)\, e^{-A(t)}\, y = b(t)\, e^{-A(t)}.$$

This exponential was chosen because of the property

$$-a(t)\, e^{-A(t)} = \big[e^{-A(t)}\big]',$$

since $A'(t) = a(t)$. Introducing this property into the differential equation,

$$e^{-A(t)} y' + \big[e^{-A(t)}\big]'\, y = b(t)\, e^{-A(t)}.$$

The product rule for derivatives says that the left-hand side above is a total derivative,

$$\big[e^{-A(t)} y\big]' = b(t)\, e^{-A(t)}.$$

One way to proceed at this point is to rewrite the right-hand side above in terms of its primitive function, $B(t) = \int e^{-A(t)}\, b(t)\, dt$, that is,

$$\big[e^{-A(t)} y\big]' = B'(t) \quad \Leftrightarrow \quad \big[e^{-A(t)} y - B(t)\big]' = 0.$$

As in the constant coefficient case, the *whole differential equation has been rewritten as the total derivative of a function*, in this case,

$$\psi(t, y(t)) = e^{-A(t)}\, y(t) - \int e^{-A(t)}\, b(t)\, dt,$$

called ***potential function***. The differential equation now has the form

$$\frac{d\psi}{dt}(t, y(t)) = 0.$$

It is simple to integrate the differential equation when written using a potential function,

$$\psi(t, y(t)) = c \quad \Leftrightarrow \quad e^{-A(t)}\, y(t) - \int e^{-A(t)}\, b(t)\, dt = c.$$

For each value of the constant $c \in \mathbb{R}$ we have the solution

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt.$$

This establishes the Theorem. □

Lemma 1.1.3 can be generalized to the variable coefficient case, henceforth stating that the integrating factor $\mu(t) = e^{-A(t)}$ describes all possible integrating factors for Eq. (1.2.1).

Lemma 1.2.2 (Integrating Factor). *Given any differentiable function y and integrable function a, every function μ satisfying*

$$(y' - ay)\mu = (y\mu)',$$

must be given in terms of any primitive of function a, $A(t) = \int a(t)\, dt$, as follows,

$$\mu(t) = e^{-A(t)}.$$

Proof of Lemma 1.2.2: Multiply Eq. (1.2.1) by a non-vanishing, but otherwise arbitrary, function with values $\mu(t)$,

$$\mu(y' - a(t)\, y) = b(t)\, \mu. \tag{1.2.3}$$

The key idea of the proof is to choose the function μ such that the following equation holds

$$\mu(y' - a(t)\, y) = \big(\mu y\big)'. \tag{1.2.4}$$

Eq. (1.2.4) is an equation for μ, since it can be rewritten as follows,

$$\mu\, y' - a(t)\, \mu y = \mu'\, y + \mu\, y' \quad \Leftrightarrow \quad -a(t)\, \mu y = \mu'\, y \quad \Leftrightarrow \quad -a(t)\, \mu = \mu'.$$

The function y does not appear the last equation above, so it does not appear in Eq. (1.2.4) either. The equation on the far right above can be solved for μ as follows,

$$\frac{\mu'}{\mu} = -a(t) \quad \Leftrightarrow \quad \big[\ln(\mu)\big]' = -a(t).$$

Integrate the equation above and denote $A(t) = \int a(t)\, dt$, so A is any primitive of a,

$$\ln(\mu) = -A(t) \quad \Leftrightarrow \quad e^{\ln(\mu)} = e^{-A(t)} \quad \Leftrightarrow \quad \mu(t) = e^{-A(t)}.$$

This establishes the Lemma. □

EXAMPLE 1.2.1: Find all solutions y to the differential equation

$$y' = \frac{3}{t}\,y + t^5.$$

SOLUTION: Rewrite the equation as

$$y' - \frac{3}{t}\,y = t^5.$$

Introduce a primitive of the coefficient function $a(t) = 3/t$,

$$A(t) = \int \frac{3}{t}\,dt = 3\ln(t) = \ln(t^3),$$

so we have $A(t) = \ln(t^3)$. The integrating factor μ is then

$$\mu(t) = e^{-A(t)} = e^{-\ln(t^3)} = e^{\ln(t^{-3})} = t^{-3},$$

hence $\mu(t) = t^{-3}$. We multiply the differential equation by the integrating factor

$$t^{-3}\big(y' - \frac{3}{t}\,y - t^5\big) = 0 \quad \Leftrightarrow \quad t^{-3}\,y' - 3\,t^{-4}\,y - t^2 = 0.$$

Since $-3\,t^{-4} = (t^{-3})'$, we get

$$t^{-3}\,y' + (t^{-3})'\,y - t^2 = 0 \quad \Leftrightarrow \quad \Big(t^{-3}\,y - \frac{t^3}{3}\Big)' = 0.$$

The potential function in this case is $\psi(t,y) = t^{-3}\,y - \dfrac{t^3}{3}$. Integrating the total derivative we obtain

$$t^{-3}\,y - \frac{t^3}{3} = c \quad \Rightarrow \quad t^{-3}\,y = c + \frac{t^3}{3},$$

so all solutions to the differential equation are $y(t) = c\,t^3 + \dfrac{t^6}{3}$, with $c \in \mathbb{R}$. ◁

1.2.2. **The Initial Value Problem.** We now generalize Theorems 1.1.5 from constant coefficients to variable coefficients equations. We first introduce the initial value problem for a variable coefficients equation, a simple generalization of Def. 1.1.4.

Definition 1.2.3. *The* ***initial value problem (IVP)*** *for a first order linear ODE is the following: Given functions $a, b : \mathbb{R} \to \mathbb{R}$ and constants t_0, $y_0 \in \mathbb{R}$, find $y : \mathbb{R} \to \mathbb{R}$ solution of*

$$y' = a(t)\,y + b(t), \qquad y(t_0) = y_0. \tag{1.2.5}$$

As we did with the constant coefficients IVP, the second equation in (1.2.5) is called the initial condition of the problem. We saw in Theorem 1.2.1 that the differential equation in (1.2.5) has infinitely many solutions, parametrized by a real constant c. The associated initial value problem has a unique solution though, because the initial condition fixes the constant c.

Theorem 1.2.4 (Variable coefficients IVP). *Given continuous functions a, b, with domain (t_1, t_2), and constants $t_0 \in (t_1, t_2)$ and $y_0 \in \mathbb{R}$, the initial value problem*

$$y' = a(t)\,y + b(t), \qquad y(t_0) = y_0, \tag{1.2.6}$$

has the unique solution $y : (t_1, t_2) \to \mathbb{R}$ given by

$$y(t) = e^{A(t)}\Big(y_0 + \int_{t_0}^{t} e^{-A(s)}\,b(s)\,ds\Big), \tag{1.2.7}$$

where the function $A(t) = \displaystyle\int_{t_0}^{t} a(s)\,ds$ is a particular primitive of function a.

Remark: In the particular case of a constant coefficients equation, that is, $a, b \in \mathbb{R}$, the solution given in Eq. (1.2.7) reduces to the one given in Eq. (1.1.10). Indeed,

$$A(t) = -\int_{t_0}^{t} a\,ds = -a(t - t_0), \qquad \int_{t_0}^{t} e^{-a(s-t_0)}\, b\,ds = -\frac{b}{a}\, e^{-a(t-t_0)} + \frac{b}{a}.$$

Therefore, the solution y can be written as

$$y(t) = y_0\, e^{a(t-t_0)} + \Big(-\frac{b}{a}\, e^{-a(t-t_0)} + \frac{b}{a}\Big)\, e^{a(t-t_0)} = \Big(y_0 + \frac{b}{a}\Big)\, e^{a(t-t_0)} - \frac{b}{a}.$$

Proof Theorem 1.2.4: We follow closely the proof of Theorem 1.1.2. From Theorem 1.2.1 we know that all solutions to the differential equation in (1.2.6) are given by

$$y(t) = c\, e^{A(t)} + e^{A(t)} \int e^{-A(t)}\, b(t)\, dt,$$

for every $c \in \mathbb{R}$. Let us use again the notation $B(t) = \int e^{-A(t)}\, b(t)\, dt$, and then introduce the initial condition in (1.2.6), which fixes the constant c,

$$y_0 = y(t_0) = c\, e^{A(t_0)} + e^{A(t_0)} B(t_0).$$

So we get the constant c,

$$c = y_0\, e^{-A(t_0)} - B(t_0).$$

Using this expression in the general solution above,

$$y(t) = \Big(y_0\, e^{-A(t_0)} - B(t_0)\Big)\, e^{A(t)} + e^{A(t)}\, B(t) = y_0\, e^{A(t)-A(t_0)} + e^{A(t)}\big(B(t) - B(t_0)\big).$$

Let us introduce the particular primitives $\hat{A}(t) = A(t) - A(t_0)$ and $\hat{B}(t) = B(t) - B(t_0)$, which vanish at t_0, that is,

$$\hat{A}(t) = \int_{t_0}^{t} a(s)\, ds, \qquad \hat{B}(t) = \int_{t_0}^{t} e^{-A(s)}\, b(s)\, ds.$$

Then the solution y of the IVP has the form

$$y(t) = y_0\, e^{\hat{A}(t)} + e^{A(t)} \int_{t_0}^{t} e^{-A(s)}\, b(s)\, ds$$

which is equivalent to

$$y(t) = y_0\, e^{\hat{A}(t)} + e^{A(t)-A(t_0)} \int_{t_0}^{t} e^{-[A(s)-A(t_0)]}\, b(s)\, ds,$$

so we conclude that

$$y(t) = e^{\hat{A}(t)} \Big(y_0 + \int_{t_0}^{t} e^{-\hat{A}(s)}\, b(s)\, ds\Big).$$

Renaming the particular primitive $\hat{A}$ simply by A, we then establish the Theorem. □

We solve the next Example following the main steps in the proof of Theorem 1.2.4 above.

Example 1.2.2: Find the function y solution of the initial value problem

$$ty' + 2y = 4t^2, \qquad y(1) = 2.$$

Solution: We first write the equation above in a way it is simple to see the functions a and b in Theorem 1.2.4. In this case we obtain

$$y' = -\frac{2}{t}\, y + 4t \quad \Leftrightarrow \quad a(t) = -\frac{2}{t}, \quad b(t) = 4t. \tag{1.2.8}$$

Now rewrite the equation as

$$y' + \frac{2}{t}\, y = 4t,$$

and multiply it by the function $\mu = e^{A(t)}$,

$$e^{A(t)}\, y' + \frac{2}{t}\, e^{A(t)}\, y = 4t\, e^{A(t)}.$$

The function A in the integrating factor $e^{A(t)}$ must be the one satisfying

$$A'(t) = \frac{2}{t} \quad \Leftrightarrow \quad A(t) = \int \frac{2}{t}\, dt.$$

In this case the differential equation can be written as

$$e^{A(t)}\, y' + A'(t)\, e^{A(t)}\, y = 4t\, e^{A(t)} \quad \Leftrightarrow \quad \big[e^{A(t)}\, y\big]' = 4t\, e^{A(t)}.$$

We now compute the function A,

$$A(t) = 2 \int \frac{dt}{t} = 2\, \ln(|t|) \quad \Rightarrow \quad A(t) = \ln(t^2).$$

This implies that

$$e^{A(t)} = t^2.$$

The differential equation, therefore, can be written as

$$\big(t^2\, y\big)' = 4t^3.$$

Integrating on both sides we obtain that

$$t^2\, y = t^4 + c \quad \Rightarrow \quad y(t) = t^2 + \frac{c}{t^2}.$$

The initial condition implies that

$$2 = y(1) = c + 1 \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad y(t) = \frac{1}{t^2} + t^2.$$

◁

Remark: It is not needed to compute the potential function to find the solution in the Example above. However, it could be useful to see this function for the differential equation in the Example. When the equation is written as

$$\big(t^2\, y\big)' = 4t^3 \quad \Leftrightarrow \quad (t^2 y)' = (t^4)' \quad \Leftrightarrow \quad (t^2 y - t^4)' = 0,$$

it is simple to see that the potential function is

$$\psi(t, y(t)) = t^2\, y(t) - t^4.$$

The differential equation is then trivial to integrate,

$$t^2 y - t^4 = c \quad \Leftrightarrow \quad t^2 y = c + t^4 \quad \Leftrightarrow \quad y(t) = \frac{c}{t^2} + t^2.$$

EXAMPLE 1.2.3: Find the solution of the problem given in Example 1.2.2 using the results of Theorem 1.2.4.

SOLUTION: We find the solution simply by using Eq. (1.2.7). First, find the integrating factor function μ as follows:

$$A(t) = -\int_1^t \frac{2}{s}\, ds = -2\big[\ln(t) - \ln(1)\big] = -2\ln(t) \quad \Rightarrow \quad A(t) = \ln(t^{-2}).$$

The integrating factor is $\mu(t) = e^{-A(t)}$, that is,

$$\mu(t) = e^{-\ln(t^{-2})} = e^{\ln(t^2)} \quad \Rightarrow \quad \mu(t) = t^2.$$

Then, we compute the solution as follows:

$$y(t) = \frac{1}{t^2}\Big[2 + \int_1^2 s^2\, 4s\, ds\Big]$$
$$= \frac{2}{t^2} + \frac{1}{t^2}\int_1^t 4s^3 ds$$
$$= \frac{2}{t^2} + \frac{1}{t^2}(t^4 - 1)$$
$$= \frac{2}{t^2} + t^2 - \frac{1}{t^2} \quad \Rightarrow \quad y(t) = \frac{1}{t^2} + t^2.$$

◁

1.2.3. **The Bernoulli Equation.** In 1696 Jacob Bernoulli struggled for months trying to solve a particular differential equation, now known as Bernoulli's differential equation. He could not solve it, so he organized a contest among his peers to solve the equation. In short time his brother Johann Bernoulli solved it. This was bad news for Jacob because the relation between the brothers was not the best at that time. Later on the equation was solved by Leibniz using a different method than Johann. Leibniz transformed the original nonlinear equation into a linear equation. We now explain Leibniz's idea in more detail.

Definition 1.2.5. *A **Bernoulli equation** in the unknown function y, determined by the functions p, $q : (t_1, t_2) \to \mathbb{R}$ and a number $n \in \mathbb{R}$, is the differential equation*

$$y' = p(t)\, y + q(t)\, y^n. \tag{1.2.9}$$

In the case that $n = 0$ or $n = 1$ the Bernoulli equation reduces to a linear equation. The interesting cases are when the Bernoulli equation is nonlinear. We now show in an Example the main idea to solve a Bernoulli equation: To transform the original nonlinear equation into a linear equation.

EXAMPLE 1.2.4: Find every solution of the differential equation

$$y' = y + 2y^5.$$

SOLUTION: This is a Bernoulli equation for $n = 5$. Divide the equation by the nonlinear factor y^5,

$$\frac{y'}{y^5} = \frac{1}{y^4} + 2.$$

Introduce the function $v = 1/y^4$ and its derivative $v' = -4(y'/y^5)$, into the differential equation above,

$$-\frac{v'}{4} = v + 2 \quad \Rightarrow \quad v' = -4\,v - 8 \quad \Rightarrow \quad v' + 4\,v = -8.$$

The last equation is a linear differential equation for the function v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{4t}$, then

$$\big(e^{4t}v\big)' = -8\,e^{4t} \quad \Rightarrow \quad e^{4t}v = -\frac{8}{4}\,e^{4t} + c.$$

We obtain that $v = c\,e^{-4t} - 2$. Since $v = 1/y^4$,

$$\frac{1}{y^4} = c\,e^{-4t} - 2 \quad \Rightarrow \quad y(t) = \pm\frac{1}{\big(c\,e^{-4t} - 2\big)^{1/4}}.$$

◁

The following result summarizes the first part of the calculation in the Example above. The nonlinear Bernoulli equation for y can be transformed into a linear equation for the function v.

Theorem 1.2.6 (Bernoulli). *The function y is a solution of the Bernoulli equation*

$$y' = p(t)\,y + q(t)\,y^n, \qquad n \neq 1,$$

iff the function $v = 1/y^{(n-1)}$ is solution of the linear differential equation

$$v' = -(n-1)p(t)\,v - (n-1)q(t).$$

This result says how to transform the Bernoulli equation for y, which is nonlinear, into a linear equation for $v = 1/y^{(n-1)}$. One then solves the linear equation for v using the integrating factor method. The last step is to transform back to $y = (1/v)^{1/(n-1)}$.

Proof of Theorem 1.2.6: Divide the Bernoulli equation by y^n,

$$\frac{y'}{y^n} = \frac{p(t)}{y^{n-1}} + q(t).$$

Introduce the new unknown $v = y^{-(n-1)}$ and compute its derivative,

$$v' = \left[y^{-(n-1)}\right]' = -(n-1)y^{-n}\,y' \quad \Rightarrow \quad -\frac{v'(t)}{(n-1)} = \frac{y'(t)}{y^n(t)}.$$

If we substitute v and this last equation into the Bernoulli equation we get

$$-\frac{v'}{(n-1)} = p(t)\,v + q(t) \quad \Rightarrow \quad v' = -(n-1)p(t)\,v - (n-1)q(t).$$

This establishes the Theorem. □

EXAMPLE 1.2.5: Given any constants a_0, b_0, find every solution of the differential equation

$$y' = a_0 y + b_0 y^3.$$

SOLUTION: This is a Bernoulli equation. Divide the equation by y^3,

$$\frac{y'}{y^3} = \frac{a_0}{y^2} + b_0.$$

Introduce the function $v = 1/y^2$ and its derivative $v' = -2(y'/y^3)$, into the differential equation above,

$$-\frac{v'}{2} = a_0 v + b_0 \quad \Rightarrow \quad v' = -2a_0 v - 2b_0 \quad \Rightarrow \quad v' + 2a_0 v = -2b_0.$$

The last equation is a linear differential equation for v. This equation can be solved using the integrating factor method. Multiply the equation by $\mu(t) = e^{2a_0 t}$,

$$\left(e^{2a_0 t} v\right)' = -2b_0\, e^{2a_0 t} \quad \Rightarrow \quad e^{2a_0 t} v = -\frac{b_0}{a_0}\, e^{2a_0 t} + c$$

We obtain that $v = c\,e^{-2a_0 t} - \dfrac{b_0}{a_0}$. Since $v = 1/y^2$,

$$\frac{1}{y^2} = c\,e^{-2a_0 t} - \frac{b_0}{a_0} \quad \Rightarrow \quad y(t) = \pm \frac{1}{\left(c\,e^{-2a_0 t} - \frac{b_0}{a_0}\right)^{1/2}}.$$

◁

Example 1.2.6: Find every solution of the equation $t\,y' = 3y + t^5\,y^{1/3}$.

Solution: Rewrite the differential equation as

$$y' = \frac{3}{t}\,y + t^4\,y^{1/3}.$$

This is a Bernoulli equation for $n = 1/3$. Divide the equation by $y^{1/3}$,

$$\frac{y'}{y^{1/3}} = \frac{3}{t}\,y^{2/3} + t^4.$$

Define the new unknown function $v = 1/y^{(n-1)}$, that is, $v = y^{2/3}$, compute is derivative, $v' = \dfrac{2}{3}\,\dfrac{y'}{y^{1/3}}$, and introduce them in the differential equation,

$$\frac{3}{2}\,v' = \frac{3}{t}\,v + t^4 \quad \Rightarrow \quad v' - \frac{2}{t}\,v = \frac{2}{3}\,t^4.$$

This is a linear equation for v. Integrate this equation using the integrating factor method. To compute the integrating factor we need to find

$$A(t) = \int \frac{2}{t}\,dt = 2\ln(t) = \ln(t^2).$$

Then, the integrating factor is $\mu(t) = e^{-A(t)}$. In this case we get

$$\mu(t) = e^{-\ln(t^2)} = e^{\ln(t^{-2})} \quad \Rightarrow \quad \mu(t) = \frac{1}{t^2}.$$

Therefore, the equation for v can be written as a total derivative,

$$\frac{1}{t^2}\Big(v' - \frac{2}{t}\,v\Big) = \frac{2}{3}\,t^2 \quad \Rightarrow \quad \Big(\frac{v}{t^2} - \frac{2}{9}\,t^3\Big)' = 0.$$

The potential function is $\psi(t,v) = v/t^2 - (2/9)t^3$ and the solution of the differential equation is $\psi(t, v(t)) = c$, that is,

$$\frac{v}{t^2} - \frac{2}{9}\,t^3 = c \quad \Rightarrow \quad v(t) = t^2\Big(c + \frac{2}{9}\,t^3\Big) \quad \Rightarrow \quad v(t) = c\,t^2 + \frac{2}{9}\,t^5.$$

Once v is known we compute the original unknown $y = \pm v^{3/2}$, where the double sign is related to taking the square root. We finally obtain

$$y(t) = \pm\Big(c\,t^2 + \frac{2}{9}\,t^5\Big)^{3/2}.$$

◁

1.2.4. **Exercises.**

1.2.1.- Find the solution y to the IVP
$$y' = -y + e^{-2t}, \quad y(0) = 3.$$

1.2.2.- Find the solution y to the IVP
$$y' = y + 2te^{2t}, \quad y(0) = 0.$$

1.2.3.- Find the solution y to the IVP
$$t\,y' + 2\,y = \frac{\sin(t)}{t}, \quad y\Big(\frac{\pi}{2}\Big) = \frac{2}{\pi},$$
for $t > 0$.

1.2.4.- Find all solutions y to the ODE
$$\frac{y'}{(t^2+1)y} = 4t.$$

1.2.5.- Find all solutions y to the ODE
$$ty' + n\,y = t^2,$$
with n a positive integer.

1.2.6.- Find all solutions to the ODE
$$2ty - y' = 0.$$
Show that given two solutions y_1 and y_2 of the equation above, the addition $y_1 + y_2$ is also a solution.

1.2.7.- Find every solution of the equation
$$y' + t\,y = t\,y^2.$$

1.2.8.- Find every solution of the equation
$$y' = -x\,y = 6x\,\sqrt{y}.$$

1.3. Separable Equations

1.3.1. **Separable Equations.** Often non-linear differential equations are more complicated to solve than the linear ones. One type of non-linear differential equations, however, is simpler to solve than linear equations. We are talking about separable equations, which are solved just by integrating on both sides of the differential equation. Precisely the first idea we had to solve a linear equation, idea that did not work in that case.

Definition 1.3.1. *A **separable** differential equation for the unknown y has the form*

$$h(y)\, y'(t) = g(t),$$

where h, g are given scalar functions.

It is not difficult to see that a differential equation $y'(t) = f(t, y(t))$ is separable iff

$$y' = \frac{g(t)}{h(y)} \quad \Leftrightarrow \quad f(t,y) = \frac{g(t)}{h(y)}.$$

Example 1.3.1:

(a) The differential equation

$$y'(t) = \frac{t^2}{1 - y^2(t)}$$

is separable, since it is equivalent to

$$\big(1 - y^2\big)\, y'(t) = t^2 \quad \Rightarrow \quad \begin{cases} g(t) = t^2, \\ h(y) = 1 - y^2. \end{cases}$$

(b) The differential equation

$$y'(t) + y^2(t)\, \cos(2t) = 0$$

is separable, since it is equivalent to

$$\frac{1}{y^2}\, y'(t) = -\cos(2t) \quad \Rightarrow \quad \begin{cases} g(t) = -\cos(2t), \\ h(y) = \dfrac{1}{y^2}. \end{cases}$$

The functions g and h are not uniquely defined; another choice in this example is:

$$g(t) = \cos(2t), \quad h(y) = -\frac{1}{y^2}.$$

(c) The linear differential equation $y'(t) = -a(t)\, y(t)$ is separable, since it is equivalent to

$$\frac{1}{y}\, y'(t) = -a(t) \quad \Rightarrow \quad \begin{cases} g(t) = -a(t), \\ h(y) = \dfrac{1}{y}. \end{cases}$$

(d) The constant coefficients linear differential equation $y'(t) = -a_0\, y(t) + b_0$ is separable, since it is equivalent to

$$\frac{1}{(-a_0\, y + b_0)}\, y'(t) = 1 \quad \Rightarrow \quad \begin{cases} g(t) = 1, \\ h(y) = \dfrac{1}{(-a_0\, y + b_0)}. \end{cases}$$

(e) The differential equation $y'(t) = e^{y(t)} + \cos(t)$ is not separable.

(f) The linear differential equation $y'(t) = -a(t)\, y(t) + b(t)$, with $b(t)$ non-constant, is not separable.

◁

The last example above shows that a linear differential equation is separable in the case that the function b is constant. So, solutions to constant coefficient linear equations can be computed using either the integrating factor method studied in Sect. 1.2 or the result we show below. Here is how we solve any separable differential equation.

Theorem 1.3.2 (Separable equations). *If the functions h, g are continuous, with $h \neq 0$, then, the separable differential equation*

$$h(y)\, y' = g(t) \tag{1.3.1}$$

has infinitely many solutions y satisfying the algebraic equation

$$H(y(t)) = G(t) + c, \tag{1.3.2}$$

where $c \in \mathbb{R}$ is arbitrary, H is a primitives (antiderivatives) of h, and G is a primitive of g.

Remark: That function H is a primitive of function h means $H' = h$. The prime here means $H' = dH/dy$. A similar relation holds for G and g, that is $G' = g$. The prime here means $G' = dG/dt$.

Before we prove this Theorem we solve a particular example. The example will help us identify the functions h, g, H and G, and it will also show how to prove the theorem.

EXAMPLE 1.3.2: Find all solutions y to the differential equation

$$y'(t) = \frac{t^2}{1 - y^2(t)}. \tag{1.3.3}$$

SOLUTION: We write the differential equation in (1.3.3) in the form $h(y)\, y' = g(t)$,

$$\big[1 - y^2(t)\big]\, y'(t) = t^2.$$

In this example the functions h and g defined in Theorem 1.3.2 are given by

$$h(y) = (1 - y^2), \qquad g(t) = t^2.$$

We now integrate with respect to t on both sides of the differential equation,

$$\int \big[1 - y^2(t)\big]\, y'(t)\, dt = \int t^2\, dt + c,$$

where c is any constant. The integral on the right-hand side can be computed explicitly. The integral on the left-hand side can be done by substitution. The substitution is

$$u = y(t), \qquad du = y'(t)\, dt.$$

This substitution on the left-hand side integral above gives,

$$\int (1 - u^2)\, du = \int t^2\, dt + c \quad \Leftrightarrow \quad u - \frac{u^3}{3} = \frac{t^3}{3} + c.$$

Substitute back the original unknown y into the last expression above and we obtain

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c.$$

We have solved the differential equation, since there are no derivatives in this last equation. When the solution is given in terms of an algebraic equation, we say that the solution y is given in implicit form. ◁

Remark: A primitive of function $h(y) = 1 - y^2$ is function $H(y) = y - y^3/3$. A primitive of function $g(t) = t^2$ is function $G(t) = t^3/3$. The implicit form of the solution found in Example 1.3.2 can be written in terms of H and G as follows,

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c. \quad \Leftrightarrow \quad H(y) = G(t) + c.$$

The expression above using H and G is the one we use in Theorem 1.3.2.

Definition 1.3.3. *A solution y of a separable equation $h(y)\,y' = g(t)$ is given in* ***implicit form*** *iff the function y is solution of the algebraic equation*

$$H\big(y(t)\big) = G(t) + c,$$

where H and G are any primitives of h and g. In the case that function H is invertible, the solution y above is given in ***explicit form*** *iff is written as*

$$y(t) = H^{-1}\big(G(t) + c\big).$$

Sometimes is difficult to find the inverse of function H. This is the case in Example 1.3.2. In such cases we leave the solution y written in implicit form. If H^{-1} is simple to compute, we write the solution y in explicit form. We now show a proof of Theorem 1.3.2 that is based in an integration by substitution, just like we did in the Example 1.3.2.

Proof of Theorem 1.3.2: Integrate with respect to t on both sides in Eq. (1.3.1),

$$h(y(t))\,y'(t) = g(t) \quad \Rightarrow \quad \int h(y(t))\,y'(t)\,dt = \int g(t)\,dt + c,$$

where c is an arbitrary constant. Introduce on the left-hand side of the second equation above the substitution

$$u = y(t), \qquad du = y'(t)\,dt.$$

The result of the substitution is

$$\int h(y(t))\,y'(t)\,dt = \int h(u)du \quad \Rightarrow \quad \int h(u)\,du = \int g(t)\,dt + c.$$

To integrate on each side of this equation means to find a function H, primitive of h, and a function G, primitive of g. Using this notation we write

$$H(u) = \int h(u)\,du, \qquad G(t) = \int g(t)\,dt.$$

Then the equation above can be written as follows,

$$H(u) = G(t) + c.$$

Substitute u back by $y(t)$. We arrive to the algebraic equation for the function y,

$$H\big(y(t)\big) = G(t) + c.$$

This establishes the Theorem. □

In the Example below we solve the same problem than in Example 1.3.2 but now we just use the result of Theorem 1.3.2.

EXAMPLE 1.3.3: Use the formula in Theorem 1.3.2 to find all solutions y to the equation

$$y'(t) = \frac{t^2}{1 - y^2(t)}. \tag{1.3.4}$$

SOLUTION: Theorem 1.3.2 tell us how to obtain the solution y. Writing Eq. (1.3.3) as

$$\big(1 - y^2\big)\,y'(t) = t^2,$$

we see that the functions h, g are given by

$$h(u) = 1 - u^2, \qquad g(t) = t^2.$$

Their primitive functions, H and G, respectively, are simple to compute,

$$h(u) = 1 - u^2 \quad \Rightarrow \quad H(u) = u - \frac{u^3}{3},$$

$$g(t) = t^2 \quad \Rightarrow \quad G(t) = \frac{t^3}{3}.$$

Then, Theorem 1.3.2 implies that the solution y satisfies the algebraic equation

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c, \tag{1.3.5}$$

where $c \in \mathbb{R}$ is arbitrary. ◁

Remark: For me it is easier to remember ideas than formulas. So for me it is easier to solve a separable equation as we did in Example 1.3.2 than in Example 1.3.3. (Although in the case of separable equations both methods are very close.)

In the next Example we show that an initial value problem can be solved even when the solutions of the differential equation are given in implicit form.

EXAMPLE 1.3.4: Find the solution of the initial value problem

$$y'(t) = \frac{t^2}{1 - y^2(t)}, \qquad y(0) = 1. \tag{1.3.6}$$

SOLUTION: From Example 1.3.2 we know that all solutions to the differential equation in (1.3.6) are given by

$$y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + c,$$

where $c \in \mathbb{R}$ is arbitrary. This constant c is now fixed with the initial condition in Eq. (1.3.6)

$$y(0) - \frac{y^3(0)}{3} = \frac{0}{3} + c \quad \Rightarrow \quad 1 - \frac{1}{3} = c \quad \Leftrightarrow \quad c = \frac{2}{3} \quad \Rightarrow \quad y(t) - \frac{y^3(t)}{3} = \frac{t^3}{3} + \frac{2}{3}.$$

So we can rewrite the algebraic equation defining the solution functions y as the roots of a cubic polynomial,

$$y^3(t) - 3y(t) + t^3 + 2 = 0.$$

◁

We present now a few more Examples.

EXAMPLE 1.3.5: Find the solution of the initial value problem

$$y'(t) + y^2(t)\cos(2t) = 0, \qquad y(0) = 1. \tag{1.3.7}$$

SOLUTION: The differential equation above is separable, with

$$g(t) = -\cos(2t), \qquad h(y) = \frac{1}{y^2},$$

therefore, it can be integrated as follows:

$$\frac{y'(t)}{y^2(t)} = -\cos(2t) \quad \Leftrightarrow \quad \int \frac{y'(t)}{y^2(t)}\,dt = -\int \cos(2t)\,dt + c.$$

Again the substitution

$$u = y(t), \qquad du = y'(t)\,dt$$

implies that

$$\int \frac{du}{u^2} = -\int \cos(2t)\, dt + c \quad \Leftrightarrow \quad -\frac{1}{u} = -\frac{1}{2}\sin(2t) + c.$$

Substitute the unknown function y back in the equation above,

$$-\frac{1}{y(t)} = -\frac{1}{2}\sin(2t) + c.$$

The solution is given in implicit form. However, in this case is simple to solve this algebraic equation for y and we obtain the following explicit form for the solutions,

$$y(t) = \frac{2}{\sin(2t) - 2c}.$$

The initial condition implies that

$$1 = y(0) = \frac{2}{0 - 2c} \quad \Leftrightarrow \quad c = -1.$$

So, the solution to the IVP is given in explicit form by

$$y(t) = \frac{2}{\sin(2t) + 2}.$$

$\lhd$

Example 1.3.6: Follow the proof in Theorem 1.3.2 to find all solutions y of the ODE

$$y'(t) = \frac{4t - t^3}{4 + y^3(t)}.$$

Solution: The differential equation above is separable, with

$$g(t) = 4t - t^3, \qquad h(y) = 4 + y^3,$$

therefore, it can be integrated as follows:

$$\big[4 + y^3(t)\big]\, y'(t) = 4t - t^3 \quad \Leftrightarrow \quad \int \big[4 + y^3(t)\big]\, y'(t)\, dt = \int (4t - t^3)\, dt + c.$$

Again the substitution

$$u = y(t), \qquad du = y'(t)\, dt$$

implies that

$$\int (4 + u^3)\, du = \int (4t - t^3)\, dt + c_0. \quad \Leftrightarrow \quad 4u + \frac{u^4}{4} = 2t^2 - \frac{t^4}{4} + c_0.$$

Substitute the unknown function y back in the equation above and calling $c_1 = 4c_0$ we obtain the following implicit form for the solution,

$$y^4(t) + 16y(t) - 8t^2 + t^4 = c_1.$$

$\lhd$

Example 1.3.7: Find the explicit form of the solution to the initial value problem

$$y'(t) = \frac{2 - t}{1 + y(t)} \qquad y(0) = 1. \tag{1.3.8}$$

Solution: The differential equation above is separable with

$$g(t) = 2 - t, \qquad h(u) = 1 + u.$$

Their primitives are respectively given by,

$$g(t) = 2 - t \quad \Rightarrow \quad G(t) = 2t - \frac{t^2}{2}, h(u) = 1 + u \quad \Rightarrow \quad H(u) = u + \frac{u^2}{2}.$$

Therefore, the implicit form of all solutions y to the ODE above are given by

$$y(t) + \frac{y^2(t)}{2} = 2t - \frac{t^2}{2} + c,$$

with $c \in \mathbb{R}$. The initial condition in Eq. (1.3.8) fixes the value of constant c, as follows,

$$y(0) + \frac{y^2(0)}{2} = 0 + c \quad \Rightarrow \quad 1 + \frac{1}{2} = c \quad \Rightarrow \quad c = \frac{3}{2}.$$

We conclude that the implicit form of the solution y is given by

$$y(t) + \frac{y^2(t)}{2} = 2t - \frac{t^2}{2} + \frac{3}{2}, \quad \Leftrightarrow \quad y^2(t) + 2y(t) + (t^2 - 4t - 3) = 0.$$

The explicit form of the solution can be obtained realizing that $y(t)$ is a root in the quadratic polynomial above. The two roots of that polynomial are given by

$$y_{\pm}(t) = \frac{1}{2}\big[-2 \pm \sqrt{4 - 4(t^2 - 4t - 3)}\big] \quad \Leftrightarrow \quad y_{\pm}(t) = -1 \pm \sqrt{-t^2 + 4t + 4}.$$

We have obtained two functions y_+ and Y_-. However, we know that there is only one solution to the IVP. We can decide which one is the solution by evaluating them at the value $t = 0$ given in the initial condition. We obtain

$$y_+(0) = -1 + \sqrt{4} = 1,$$
$$y_-(0) = -1 - \sqrt{4} = -3.$$

Therefore, the solution is y_+, that is, the explicit form of the solution is

$$y(t) = -1 + \sqrt{-t^2 + 4t + 4}.$$

$\triangleleft$

1.3.2. **Euler Homogeneous Equations.** Sometimes a differential equation is not separable but it can be transformed into a separable equation by a change in the unknown function. This is the case for a type of differential equations called Euler homogeneous equations.

Definition 1.3.4. *A first order differential equation of the form $y'(t) = f\big(t, y(t)\big)$ is called* ***Euler homogeneous*** *iff for every real numbers t, u and every $c \neq 0$ the function f satisfies*

$$f(ct, cu) = f(t, u).$$

Remark: The condition $f(ct, cu) = f(t, u)$ means that the function f is scale invariant.

Remark: A function of two variables, f, with values $f(t, u)$, is scale invariant iff the function depends on (t, u) only through their quotient, u/t. In other words, there exists a single variable function F such that

$$f(t, u) = F\Big(\frac{u}{t}\Big).$$

Proof of the Remark:

(a) $(\Rightarrow)$ If $f(t, u) = F(u/t)$, then $f(ct, cu) = F((cu)/(ct)) = F(u/t) = f(u, t)$.
(b) $(\Leftarrow)$ If $f(t, u) = f(ct, cu)$ then pick $c = 1/t$, which gives $f(t, u) = f(t/t, u/t) = f(1, u/t)$; denoting $F(u/t) = f(1, u/t)$ we conclude that $f(t, u) = F(u/t)$.

This establishes the Remark. $\square$

From the previous two remarks we conclude that a first order differential equation is Euler homogeneous iff it has the form

$$y'(t) = F\Big(\frac{y(t)}{t}\Big). \tag{1.3.9}$$

Equation 1.3.9 is often in the literature the definition of an Euler homogeneous equation.

Example 1.3.8: Show that the equation below is Euler homogeneous,

$$(t-y)\,y' - 2y + 3t + \frac{y^2}{t} = 0.$$

Solution: Rewrite the equation in the standard form

$$(t-y)\,y' = 2y - 3t - \frac{y^2}{t} \quad \Rightarrow \quad y' = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t-y)}.$$

So the function f in this case is given by

$$f(t,y) = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t-y)}.$$

We now check whether f is scale invariant or not. We compute $f(ct, cy)$ and we check whether the c cancels out or not.

$$f(ct,cy) = \frac{\Big(2cy - 3ct - \dfrac{c^2y^2}{ct}\Big)}{(ct-cy)} = \frac{c\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{c(t-y)} = f(t,y).$$

We conclude that f is scale invariant, so the differential equation is Euler homogeneous.$\lhd$

Remark: We verified that the differential equation in Example 1.3.8 is Euler homogeneous. We can now rewrite it in terms of the single variable function F given in the a remark above. There are at least two ways to find that function F:

(a) Use the definition given in a remark above, $F(y/t) = f(1, y/t)$. Recall that

$$f(t,y) = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t-y)} \quad \Rightarrow \quad f(1,y) = \frac{(2y - 3 - y^2)}{(1-y)}.$$

Since $F(y/t) = f(1, y/t)$, we obtain $F(y/t) = \dfrac{2\Big(\dfrac{y}{t}\Big) - 3 - \Big(\dfrac{y}{t}\Big)^2}{\Big[1 - \Big(\dfrac{y}{t}\Big)\Big]}$.

(b) Multiply f by one, in the form $(1/t)/(1/t)$, that is,

$$f(t,y) = \frac{\Big(2y - 3t - \dfrac{y^2}{t}\Big)}{(t-y)}\,\frac{\Big(\dfrac{1}{t}\Big)}{\Big(\dfrac{1}{t}\Big)} \quad \Rightarrow \quad f(t,y) = \frac{2\Big(\dfrac{y}{t}\Big) - 3 - \Big(\dfrac{y}{t}\Big)^2}{\Big[1 - \Big(\dfrac{y}{t}\Big)\Big]}.$$

The right-hand side on the last equation above depends only on y/t. So we have shown that $f(t,y) = F(y/t)$, where

$$F(y/t) = \frac{2\Big(\dfrac{y}{t}\Big) - 3 - \Big(\dfrac{y}{t}\Big)^2}{\Big[1 - \Big(\dfrac{y}{t}\Big)\Big]}.$$

Recall that f is scale invariant, so $f(t,y) = f(1, y/t)$.

EXAMPLE 1.3.9: Determine whether the equation below is Euler homogeneous,

$$y' = \frac{t^2}{1 - y^3}.$$

SOLUTION: The differential equation is written in the standard for $y' = f(t, y)$, where

$$f(t, y) = \frac{t^2}{1 - y^3}.$$

We now check whether the function f is scale invariant.

$$f(ct, cy) = \frac{c^2t^2}{1 - c^3y^3} = \frac{c^2t^2}{c^3((1/c^3) - y^3)} = \frac{t^2}{c((1/c^3) - y^3)}.$$

Since $f(ct, cy) \neq f(t, y)$, we conclude that the equation is not Euler homogeneous. $\lhd$

Up to this point we know how to identify an Euler homogeneous differential equation. Now we say how to solve an Euler homogeneous equation.

Theorem 1.3.5 (Euler Homogeneous). *If the differential equation for a function y*

$$y'(t) = f\big(t, y(t)\big)$$

is Euler homogeneous, then the function $v(t) = \dfrac{y(t)}{t}$ satisfies the separable equation

$$\frac{v'}{(F(v) - v)} = \frac{1}{t},$$

where we have denoted $F(v) = f(1, v)$.

Remark: Theorem 1.3.5 says that Euler homogeneous equations can be transformed into separable equations. We used a similar idea to solve a Bernoulli equation, where we transformed a non-linear equation into a linear one. In the case of an Euler homogeneous equation for the function y, we transform it into a separable equation for the unknown function $v = y/t$. We solve for v in implicit or explicit form. Then, we transform back to $y = t\,v$.

Proof of Theorem 1.3.5: If $y' = f(t, y)$ is homogeneous, then we saw in one of the remarks above that the equation can be written as $y' = F(y/t)$, where $F(y/t) = f(1, y/t)$. Introduce the function $v = y/t$ into the differential equation,

$$y' = F(v).$$

We still need to replace y' in terms of v. This is done as follows,

$$y(t) = t\,v(t) \quad \Rightarrow \quad y'(t) = v(t) + t\,v'(t).$$

Introducing these expressions into the differential equation for y we get

$$v + t\,v' = F(v) \quad \Rightarrow \quad v' = \frac{\big(F(v) - v\big)}{t} \quad \Rightarrow \quad \frac{v'}{\big(F(v) - v\big)} = \frac{1}{t}.$$

The equation on the far right is separable. This establishes the Theorem. $\square$

EXAMPLE 1.3.10: Find all solutions y of the differential equation $y' = \dfrac{t^2 + 3y^2}{2ty}$.

SOLUTION: The equation is Euler homogeneous, since

$$f(ct, cy) = \frac{c^2t^2 + 3c^2y^2}{2ctcy} = \frac{c^2(t^2 + 3y^2)}{2c^2ty} = \frac{t^2 + 3y^2}{2ty} = f(t, y).$$

The next step is to compute the function F. Since we got a c^2 in numerator and denominator, we choose to multiply the right-hand side of the differential equation by one in the form $(1/t^2)/(1/t^2)$,

$$y' = \frac{(t^2 + 3y^2)}{2ty} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \quad \Rightarrow \quad y' = \frac{1 + 3\left(\frac{y}{t}\right)^2}{2\left(\frac{y}{t}\right)}.$$

Now we introduce the change of unknown $v = y/t$, so $y = t\,v$ and $y' = v + t\,v'$. Hence

$$v + t\,v' = \frac{1 + 3v^2}{2v} \quad \Rightarrow \quad t\,v' = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2 - 2v^2}{2v}$$

We obtain the separable equation $v' = \dfrac{1}{t}\left(\dfrac{1 + v^2}{2v}\right)$. We rewrite and integrate it,

$$\frac{2v}{1 + v^2}\,v' = \frac{1}{t} \quad \Rightarrow \quad \int \frac{2v}{1 + v^2}\,v'\,dt = \int \frac{1}{t}\,dt + c_0.$$

The substitution $u = 1 + v^2(t)$ implies $du = 2v(t)\,v'(t)\,dt$, so

$$\int \frac{du}{u} = \int \frac{dt}{t} + c_0 \quad \Rightarrow \quad \ln(u) = \ln(t) + c_0 \quad \Rightarrow \quad u = e^{\ln(t) + c_0}.$$

But $u = e^{\ln(t)} e^{c_0}$, so denoting $c_1 = e^{c_0}$, then $u = c_1 t$. Hence, the explicit form of the solution can be computed as follows,

$$1 + v^2 = c_1 t \quad \Rightarrow \quad 1 + \left(\frac{y}{t}\right)^2 = c_1 t \quad \Rightarrow \quad y(t) = \pm t\sqrt{c_1 t - 1}.$$

◁

Example 1.3.11: Find all solutions y of the differential equation $y' = \dfrac{t(y + 1) + (y + 1)^2}{t^2}$.

Solution: This equation is not homogeneous in the unknown y and variable t, however, it becomes homogeneous in the unknown $u(t) = y(t) + 1$ and the same variable t. Indeed, $u' = y'$, thus we obtain

$$y' = \frac{t(y + 1) + (y + 1)^2}{t^2} \quad \Leftrightarrow \quad u' = \frac{tu + u^2}{t^2} \quad \Leftrightarrow \quad u' = \frac{u}{t} + \left(\frac{u}{t}\right)^2.$$

Therefore, we introduce the new variable $v = u/t$, which satisfies $u = t\,v$ and $u' = v + t\,v'$. The differential equation for v is

$$v + t\,v' = v + v^2 \quad \Leftrightarrow \quad t\,v' = v^2 \quad \Leftrightarrow \quad \int \frac{v'}{v^2}\,dt = \int \frac{1}{t}\,dt + c,$$

with $c \in \mathbb{R}$. The substitution $w = v(t)$ implies $dw = v'\,dt$, so

$$\int w^{-2}\,dw = \int \frac{1}{t}\,dt + c \quad \Leftrightarrow \quad -w^{-1} = \ln(|t|) + c \quad \Leftrightarrow \quad w = -\frac{1}{\ln(|t|) + c}.$$

Substituting back v, u and y, we obtain $w = v(t) = u(t)/t = [y(t) + 1]/t$, so

$$\frac{y + 1}{t} = -\frac{1}{\ln(|t|) + c} \quad \Leftrightarrow \quad y(t) = -\frac{t}{\ln(|t|) + c} - 1.$$

◁

1.3.3. **Exercises.**

1.3.1.- Find all solutions y to the ODE

$$y' = \frac{t^2}{y}.$$

Express the solutions in explicit form.

1.3.2.- Find every solution y of the ODE

$$3t^2 + 4y^3 y' - 1 + y' = 0.$$

Leave the solution in implicit form.

1.3.3.- Find the solution y to the IVP

$$y' = t^2 y^2, \quad y(0) = 1.$$

1.3.4.- Find every solution y of the ODE

$$ty + \sqrt{1+t^2}\, y' = 0.$$

1.3.5.- Find every solution y of the Euler homogeneous equation

$$y' = \frac{y+t}{t}.$$

1.3.6.- Find all solutions y to the ODE

$$y' = \frac{t^2 + y^2}{ty}.$$

1.3.7.- Find the explicit solution to the IVP

$$(t^2 + 2ty)\, y' = y^2, \quad y(1) = 1.$$

1.3.8.- Prove that if $y' = f(t, y)$ is an Euler homogeneous equation and $y_1(t)$ is a solution, then $y(t) = (1/k)\, y_1(kt)$ is also a solution for every non-zero $k \in \mathbb{R}$.

1.4. Exact Equations

A differential equation is called exact when it can be written as a total derivative of an appropriate function, called potential function. When the equation is written in that way it is simple to find implicit solutions. Given an exact equation, we just need to find a potential function, and a solution of the differential equation will be determined by any level surface of that potential function.

There are differential equations that are not exact but they can be converted into exact equations when they are multiplied by an appropriate function, called an integrating factor. An integrating factor converts a non-exact equation into an exact equation. Linear differential equations are a particular case of this type of equations, and we have studied them in Sections 1.1 and 1.2. For linear equations we computed integrating factors that transformed the equation into a derivative of a potential function. We now generalize this idea to a class of non-linear equations.

1.4.1. Exact Differential Equations.

We start with a definition of an exact equation that is simple to verify in particular examples. Partial derivatives of certain functions must agree. Later on we show that this condition is equivalent to the existence of a potential function.

Definition 1.4.1. *The differential equation in the unknown function y given by*

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0$$

is called ***exact*** *on an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ iff for every point $(t, u) \in R$ the functions $M, N : R \to \mathbb{R}$ are continuously differentiable and satisfy the equation*

$$\partial_t N(t, u) = \partial_u M(t, u)$$

We use the notation for partial derivatives $\partial_t N = \dfrac{\partial N}{\partial t}$ and $\partial_u M = \dfrac{\partial M}{\partial u}$. Let us see whether the following equations are exact or not.

Example 1.4.1: Show whether the differential equation below is exact or not,

$$2ty(t)\, y'(t) + 2t + y^2(t) = 0.$$

Solution: We first identify the functions N and M. This is simple in this case, since

$$\big[2ty(t)\big]\, y'(t) + \big[2t + y^2(t)\big] = 0 \quad \Rightarrow \quad N(t, u) = 2tu, \quad M(t, u) = 2t + u^2.$$

The equation is indeed exact, since

$$\left.\begin{array}{lcl} N(t, u) = 2tu & \Rightarrow & \partial_t N(t, u) = 2u, \\ M(t, u) = 2t + u^2 & \Rightarrow & \partial_u M(t, u) = 2u, \end{array}\right\} \quad \Rightarrow \quad \partial_t N(t, u) = \partial_u M(t, u).$$

Therefore, the differential equation is exact. ◁

Example 1.4.2: Show whether the differential equation below is exact or not,

$$\sin(t) y'(t) + t^2 e^{y(t)} y'(t) - y'(t) = -y(t)\cos(t) - 2te^{y(t)}.$$

Solution: We first identify the functions N and M by rewriting the equation as follows,

$$\big[\sin(t) + t^2 e^{y(t)} - 1\big]\, y'(t) + \big[y(t)\cos(t) + 2te^{y(t)}\big] = 0$$

we can see that

$$\begin{array}{lcl} N(t, u) = \sin(t) + t^2 e^u - 1 & \Rightarrow & \partial_t N(t, u) = \cos(t) + 2te^u, \\ M(t, u) = u\cos(t) + 2te^u & \Rightarrow & \partial_u M(t, u) = \cos(t) + 2te^u. \end{array}$$

Therefore, the equation is exact, since

$$\partial_t N(t,u) = \partial_u M(t,u).$$

$\lhd$

The last example shows whether the linear differential equations we studied in Section 1.2 are exact or not.

EXAMPLE 1.4.3: Show whether the linear differential equation below is exact or not,

$$y'(t) = a(t)\,y(t) + b(t), \qquad a(t) \neq 0.$$

SOLUTION: We first find the functions N and M rewriting the equation as follows,

$$y' + a(t)y - b(t) = 0 \quad \Rightarrow \quad N(t,u) = 1, \quad M(t,u) = -a(t)\,u - b(t).$$

Now is simple to see what the outcome will be, since

$$\left.\begin{array}{lll} N(t,u) = 1 & \Rightarrow & \partial_t N(t,u) = 0, \\ M(t,u) = -a(t)u - b(t) & \Rightarrow & \partial_u M(t,u) = -a(t), \end{array}\right\} \quad \Rightarrow \quad \partial_t N(t,u) \neq \partial_u M(t,u).$$

The differential equation is not exact. $\lhd$

1.4.2. **The Poincaré Lemma.** It is simple to check if a differential equation is exact. It is not so simple, however, to write the exact equation as a total derivative. The main difficulties are to show that a potential function exists and how to relate the potential function to the differential equation. Both results were proven by Henri Poincaré around 1880. The proof is rather involved, so we show this result without the complicated part of the proof.

Lemma 1.4.2 (Poincaré). *Continuously differentiable functions $M, N : R \to \mathbb{R}$, on an open rectangle $R = (t_1, t_2) \times (u_1, u_2)$, satisfy the equation*

$$\partial_t N(t,u) = \partial_u M(t,u) \tag{1.4.1}$$

iff there exists a twice continuously differentiable function $\psi : R \to \mathbb{R}$, called potential function, such that for all $(t,u) \in R$ holds

$$\partial_u \psi(t,u) = N(t,u), \qquad \partial_t \psi(t,u) = M(t,u). \tag{1.4.2}$$

Remark: A differential equation provides the definition of functions N and M. The exact condition in (1.4.1) is equivalent to the existence of the potential function ψ, which relates to N and M through Eq. (1.4.2).

Proof of Lemma 1.4.2:
($\Leftarrow$) We assume that the potential function ψ is given and satisfies Eq. (1.4.2). Since ψ is twice continuously differentiable, its cross derivatives are the same, that is, $\partial_t \partial_u \psi = \partial_u \partial_t \psi$. We then conclude that

$$\partial_t N = \partial_t \partial_u \psi = \partial_u \partial_t \psi = \partial_u M.$$

($\Rightarrow$) It is not given. See [9]. $\square$

Remark: If a differential equation is exact, then the Poincaré Lemma says that the potential function exists for that equation. Not only that, but it gives us a way to compute the potential function by integrating the equations in (1.4.2).

Now we verify that a given function ψ is the potential function for an exact differential equation. Later on we show how to compute such potential function from the differential equation, by integrating the equations in (1.4.2).

Example 1.4.4: Show that the function $\psi(t,u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t)\, y'(t) + 2t + y^2(t) = 0.$$

Solution: In Example 1.4.1 we showed that the differential equation above is exact, since

$$N(t,u) = 2tu, \qquad M(t,u) = 2t + u^2 \quad \Rightarrow \quad \partial_t N = 2u = \partial_u M.$$

Let us check that the function $\psi(t,u) = t^2 + tu^2$, is the potential function of the differential equation. First compute the partial derivatives,

$$\partial_t \psi = 2t + u^2 = M, \qquad \partial_u \psi = 2tu = N.$$

Now use the chain rule to compute the t derivative of the following function,

$$\frac{d}{dt}\psi(t, y(t)) = \partial_y \psi\, \frac{dy}{dt} + \partial_t \psi.$$

But we have just computed these partial derivatives,

$$\frac{d}{dt}\psi(t, y(t)) = \big(2t\, y(t)\big)\, y' + \big(2t + y^2(t)\big) = 0.$$

So we have shown that the differential equation can be written as $\dfrac{d\psi}{dt}\big(t, y(t)\big) = 0$. ◁

1.4.3. **Solutions and a Geometric Interpretation.** A potential function ψ of an exact differential equation is crucial to find implicit solutions of that equation. Solutions are defined by level curves of a potential function.

Theorem 1.4.3 (Exact equation). *If the differential equation*

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0 \tag{1.4.3}$$

is exact on $R = (t_1, t_2) \times (u_1, u_2)$, then every solution y must satisfy the algebraic equation

$$\psi(t, y(t)) = c, \tag{1.4.4}$$

where $c \in \mathbb{R}$ and $\psi : R \to \mathbb{R}$ is a potential function for Eq. (1.4.3).

Proof of Theorem 1.4.3: The differential equation in (1.4.3) is exact, then Lemma 1.4.2 implies that there exists a potential function ψ satisfying Eqs. (1.4.2). Write functions N and M in the differential equation in terms of $\partial_y \psi$ and $\partial_t \psi$. The differential equation is then given by

$$\begin{aligned} 0 &= N(t, y(t))\, y'(t) + M(t, y(t)) \\ &= \partial_y \psi(t, y(t))\, \frac{d}{dt} y(t) + \partial_t \psi(t, y(t)). \end{aligned}$$

The chain rule, which is the derivative of a composition of functions, implies that

$$0 = \partial_y \psi(t, y(t))\, \frac{d}{dt} y(t) + \partial_t \psi(t, y(t)) = \frac{d}{dt}\psi(t, y(t)).$$

The differential equation has been rewritten as a total t-derivative of the potential function,

$$\frac{d}{dt}\psi(t, y(t)) = 0.$$

This equation is simple to integrate,

$$\psi(t, y(t)) = c,$$

where c is an arbitrary constant. This establishes the Theorem. □

Remark: There is a nice geometrical interpretation of both an exact differential equation and its solutions. We can start with Eq. (1.4.4), which says that a solution y is defined by a level curve of the potential function,

$$\psi = c.$$

On the one hand, a solution function y defines on the ty-plane a vector-valued function $\boldsymbol{r}(t) = \langle t, y(t)\rangle$. The t-derivative of this function is,

$$\frac{d\boldsymbol{r}}{dt} = \langle 1, y'\rangle,$$

which must be tangent to the curve defined by $\boldsymbol{r}$. On the other hand, the vector gradient of the potential function,

$$\nabla\psi = \langle \partial_t\psi, \partial_y\psi\rangle = \langle M, N\rangle.$$

must be perpendicular to the curve defined by $\boldsymbol{r}$. This is precisely what the differential equation for y is telling us, since

$$0 = Ny' + M = \langle M, N\rangle \cdot \langle 1, y'\rangle,$$

we see that the differential equation for y is equivalent to

$$\nabla\psi \cdot \frac{d\boldsymbol{r}}{dt} = 0.$$

In Fig. 2 we picture the case where the potential function is a paraboloid, $\psi(t, y) = t^2 + y^2$.

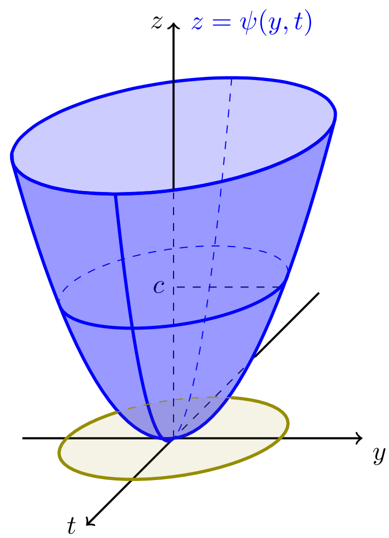


FIGURE 2. Potential ψ with level curve $\psi = c$ defines a solution y on the ty-plane.

EXAMPLE 1.4.5: Find all solutions y to the differential equation

$$2ty(t)\, y'(t) + 2t + y^2(t) = 0.$$

SOLUTION: The first step is to verify whether the differential equation is exact. We know the answer, the equation is exact, we did this calculation before in Example 1.4.1, but we reproduce it here anyway.

$$\left.\begin{array}{lcl} N(t,u) = 2tu & \Rightarrow & \partial_t N(t,u) = 2u, \\ M(t,u) = 2t + u^2 & \Rightarrow & \partial_u M(t,u) = 2u. \end{array}\right\} \quad \Rightarrow \quad \partial_t N(t,u) = \partial_u M(t,u).$$

Since the equation is exact, Lemma 1.4.2 implies that there exists a potential function ψ satisfying the equations

$$\partial_u \psi(t,u) = N(t,u), \tag{1.4.5}$$

$$\partial_t \psi(t,u) = M(t,u). \tag{1.4.6}$$

We now proceed to compute the function ψ. Integrate Eq. (1.4.5) in the variable u keeping the variable t constant,

$$\partial_u \psi(t,u) = 2tu \quad \Rightarrow \quad \psi(t,u) = \int 2tu\, du + g(t),$$

where g is a constant of integration on the variable u, so g can only depend on t. We obtain

$$\psi(t,u) = tu^2 + g(t). \tag{1.4.7}$$

Introduce into Eq. (1.4.6) the expression for the function ψ in Eq. (1.4.7) above, that is,

$$u^2 + g'(t) = \partial_t \psi(t,u) = M(t,u) = 2t + u^2 \quad \Rightarrow \quad g'(t) = 2t$$

Integrate in t the last equation above, and choose the integration constant to be zero,

$$g(t) = t^2.$$

We have found that a potential function is given by

$$\psi(t,u) = tu^2 + t^2.$$

Therefore, Theorem 1.4.3 implies that all solutions y satisfy the implicit equation

$$ty^2(t) + t^2 = c,$$

where $c \in R$ is an arbitrary constant.

Remark: The choice $g(t) = t^2 + c_0$ only modifies the constant c above. $\triangleleft$

Example 1.4.6: Find all solutions y to the equation

$$\big[\sin(t) + t^2 e^{y(t)} - 1\big]\, y'(t) + y(t)\cos(t) + 2te^{y(t)} = 0.$$

Solution: The first step is to verify whether the differential equation is exact. We know the answer, the equation is exact, we did this calculation before in Example 1.4.2, but we reproduce it here anyway.

$$N(t,u) = \sin(t) + t^2 e^u - 1 \quad \Rightarrow \quad \partial_t N(t,u) = \cos(t) + 2te^u,$$
$$M(t,u) = u\cos(t) + 2te^u \quad \Rightarrow \quad \partial_u M(t,u) = \cos(t) + 2te^u.$$

Therefore, the differential equation is exact. Then, Lemma 1.4.2 implies that there exists a potential function ψ satisfying the equations

$$\partial_u \psi(t,u) = N(t,u), \tag{1.4.8}$$
$$\partial_t \psi(t,u) = M(t,u). \tag{1.4.9}$$

We know proceed to compute the function ψ. We first integrate in the variable u the equation $\partial_u \psi = N$ keeping the variable t constant,

$$\partial_u \psi(t,u) = \sin(t) + t^2 e^u - 1 \quad \Rightarrow \quad \psi(t,u) = \int \big[\sin(t) + t^2 e^u - 1\big]\, du + g(t)$$

where g is a constant of integration on the variable u, so g can only depend on t. We obtain

$$\psi(t,u) = u\sin(t) + t^2 e^u - u + g(t).$$

Now introduce the expression above for the potential function ψ in Eq. (1.4.9), that is,

$$u\cos(t) + 2te^u + g'(t) = \partial_t \psi(t,u) = M(t,u) = u\cos(t) + 2te^u \quad \Rightarrow \quad g'(t) = 0.$$

The solution is $g(t) = c_0$, with c_0 a constant, but we can always choose that constant to be zero. (Se the Remark at the end of the previous example.) We conclude that

$$g(t) = 0.$$

We found g, so we have the complete potential function,

$$\psi(t,u) = u\sin(t) + t^2 e^u - u.$$

Theorem 1.4.3 implies that any solution y satisfies the implicit equation

$$y(t)\sin(t) + t^2 e^{y(t)} - y(t) = c.$$

The solution y above cannot be written in explicit form.

Remark: The choice $g(t) = c_0$ only modifies the constant c above. $\triangleleft$

1.4.4. **The Integrating Factor Method.** Sometimes a non-exact differential equation can be rewritten as an exact differential equation. One way this could happen is multiplying the differential equation by an appropriate function. If the new equation is exact, the multiplicative function is called an integrating factor.

This is precisely the case with linear differential equations. We have seen in Example 1.4.3 that linear equations with coefficient $a \neq 0$ are not exact. But in Section 1.2 we have obtained solutions to linear equations multiplying the equation by an appropriate function. We called that function an integrating factor. That function converted the original differential equation into a total derivative of a function, which we called potential function. Using the terminology of this Section, the integrating factor transformed a linear equation into an exact equation.

Now we generalize this idea to non-linear differential equations.

Theorem 1.4.4 (Integrating factor I). *Assume that the differential equation*

$$N(t,y)\,y' + M(t,y) = 0 \tag{1.4.10}$$

is not exact because $\partial_t N(t,u) \neq \partial_u M(t,u)$ holds for the continuously differentiable functions M, N on their domain $R = (t_1,t_2)\times(u_1,u_2)$. If $N \neq 0$ on R and the function

$$\frac{1}{N(t,u)}\big[\partial_u M(t,u) - \partial_t N(t,u)\big] \tag{1.4.11}$$

does not depend on the variable u, then the equation below is exact,

$$(\mu\, N)\,y' + (\mu\, M) = 0 \tag{1.4.12}$$

where the function μ, which depends only on $t \in (t_1,t_2)$, is a solution of the equation

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{N(t,u)}\big[\partial_u M(t,u) - \partial_t N(t,u)\big].$$

Proof of Theorem 1.4.4: We know that the original differential equation in (1.4.10) is not exact because $\partial_t N \neq \partial_u M$. Now multiply the differential equation by a non-zero function μ that depends only on t,

$$(\mu N)\,y' + (\mu M) = 0. \tag{1.4.13}$$

We look for a function μ such that this new equation is exact. This means that μ must satisfy the equation

$$\partial_t(\mu N) = \partial_u(\mu M).$$

Recalling that μ depends only on t and denoting $\partial_t \mu = \mu'$, we get

$$\mu'\,N + \mu\,\partial_t N = \mu\,\partial_u M \quad\Rightarrow\quad \mu'\,N = \mu\,(\partial_u M - \partial_t N).$$

So the differential equation in (1.4.13) is exact iff holds

$$\frac{\mu'}{\mu} = \frac{(\partial_u M - \partial_t N)}{N},$$

and a necessary condition for such an equation to have solutions is that the right-hand side be independent of the variable u. This establishes the Theorem. □

EXAMPLE 1.4.7: Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0. \tag{1.4.14}$$

SOLUTION: We first verify whether this equation is exact:

$$N(t,u) = t^2 + tu \quad\Rightarrow\quad \partial_t N(t,u) = 2t + u,$$
$$M(t,u) = 3tu + u^2 \quad\Rightarrow\quad \partial_u M(t,u) = 3t + 2u,$$

therefore, the differential equation is not exact. We now verify whether the extra condition in Theorem 1.4.4 holds, that is, whether the function in (1.4.11) is u-independent;

$$\begin{aligned}\frac{1}{N(t,u)}\big[\partial_u M(t,u) - \partial_t N(t,u)\big] &= \frac{1}{(t^2+tu)}\big[(3t+2u)-(2t+u)\big]\\ &= \frac{1}{t(t+u)}\,(t+u)\\ &= \frac{1}{t}.\end{aligned}$$

The function $(\partial_u M - \partial_t N)/N$ is u independent. Therefore, Theorem 1.4.4 implies that the differential equation in (1.4.14) can be transformed into an exact equation. We need to multiply the differential equation by a function μ solution of the equation

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{N}\big[\partial_u M - \partial_t N\big] = \frac{1}{t} \quad\Rightarrow\quad \ln(\mu(t)) = \ln(t) \quad\Rightarrow\quad \mu(t) = t,$$

where we have chosen in second equation the integration constant to be zero. Then, multiplying the original differential equation in (1.4.14) by the integrating factor μ we obtain

$$\big[3t^2\,y(t) + t\,y^2(t)\big] + \big[t^3 + t^2\,y(t)\big]\,y'(t) = 0. \tag{1.4.15}$$

This latter equation is exact, since

$$\begin{aligned}\tilde N(t,u) &= t^3 + t^2u &&\Rightarrow && \partial_t\tilde N(t,u) = 3t^2 + 2tu,\\ \tilde M(t,u) &= 3t^2u + tu^2 &&\Rightarrow && \partial_u\tilde M(t,u) = 3t^2 + 2tu,\end{aligned}$$

so we get the exactness condition $\partial_t\tilde N = \partial_u\tilde M$. The solution y can be found as we did in the previous examples in this Section. That is, we find the potential function ψ by integrating the equations

$$\partial_u\psi(t,u) = \tilde N(t,u), \tag{1.4.16}$$

$$\partial_t\psi(t,u) = \tilde M(t,u). \tag{1.4.17}$$

From the first equation above we obtain

$$\partial_u\psi = t^3 + t^2u \quad\Rightarrow\quad \psi(t,u) = \int \big(t^3 + t^2u\big)\,du + g(t).$$

Integrating on the right hand side above we arrive to

$$\psi(t,u) = t^3u + \frac{1}{2}\,t^2u^2 + g(t).$$

Introduce the expression above for ψ in Eq. (1.4.17),

$$\begin{aligned}3t^2u + tu^2 + g'(t) = \partial_t\psi(t,u) &= \tilde M(t,u) = 3t^2u + tu^2,\\ g'(t) &= 0.\end{aligned}$$

A solution to this last equation is $g(t) = 0$. So we get a potential function

$$\psi(t,u) = t^3u + \frac{1}{2}\,t^2u^2.$$

All solutions y to the differential equation in (1.4.14) satisfy the equation

$$t^3\,y(t) + \frac{1}{2}\,t^2\big[y(t)\big]^2 = c_0,$$

where $c_0 \in \mathbb{R}$ is arbitrary. ◁

We have seen in Example 1.4.3 that linear differential equations with $a \neq 0$ are not exact. In Section 1.2 we found solutions to linear equations using the integrating factor method. We multiplied the linear equation by a function that transformed the equation into a total derivative. Those calculations are now a particular case of Theorem 1.4.4, as we can see it in the following Example.

EXAMPLE 1.4.8: Use Theorem 1.4.4 to find all solutions to the linear differential equation

$$y'(t) = a(t)\, y(t) + b(t), \qquad a(t) \neq 0. \tag{1.4.18}$$

SOLUTION: We first write the linear equation in a way we can identify functions N and M,

$$y' + \big[-a(t)\, y - b(t)\big] = 0.$$

We now verify whether the linear equation is exact or not. Actually, we have seen in Example 1.4.3 that this equation is not exact, since

$$N(t,u) = 1 \qquad \Rightarrow \qquad \partial_t N(t,u) = 0,$$
$$M(t,u) = -a(t)u - b(t) \qquad \Rightarrow \qquad \partial_u M(t,u) = -a(t).$$

But now we can go further, we can check wheteher the condtion in Theorem 1.4.4 holds or not. We compute the function

$$\frac{1}{N(t,u)}\big[\partial_u M(t,u) - \partial_t N(t,u)\big] = -a(t)$$

and we see that it is independent of the variable u. Theorem 1.4.4 says that we can transform the linear equation into an exact equation. We only need to multiply the linear equation by a function μ, solution of the equation

$$\frac{\mu'(t)}{\mu(t)} = -a(t) \quad \Rightarrow \quad \mu(t) = e^{-A(t)}, \quad A(t) = \int a(t)dt.$$

This is the same integrating factor we discovered in Section 1.2. Therefore, the equation below is exact,

$$e^{-A(t)} y' - \big[a(t)\, e^{-A(t)} y - b(t)\, e^{-A(t)}\big] = 0. \tag{1.4.19}$$

This new version of the linear equation is exact, since

$$\tilde{N}(t,u) = e^{-A(t)} \qquad \Rightarrow \qquad \partial_t \tilde{N}(t,u) = -a(t)\, e^{-A(t)},$$
$$\tilde{M}(t,u) = -a(t)\, e^{-A(t)} u - b(t)\, e^{-A(t)} \qquad \Rightarrow \qquad \partial_u \tilde{M}(t,u) = -a(t)\, e^{-A(t)}.$$

Since the linear equation is now exact, the solutions y can be found as we did in the previous examples in this Section. We find the potential function ψ integrating the equations

$$\partial_u \psi(t,u) = \tilde{N}(t,u), \tag{1.4.20}$$
$$\partial_t \psi(t,u) = \tilde{M}(t,u). \tag{1.4.21}$$

From the first equation above we obtain

$$\partial_u \psi = e^{-A(t)} \quad \Rightarrow \quad \psi(t,u) = \int e^{-A(t)}\, du + g(t).$$

The integral is simple, since $e^{-A(t)}$ is u-independent. We then get

$$\psi(t,u) = e^{-A(t)} u + g(t).$$

We introduce the expression above for ψ in Eq. (1.4.17),

$$-a(t)\, e^{-A(t)} u + g'(t) = \partial_t \psi(t,u) = \tilde{M}(t,u) = -a(t)\, e^{-A(t)} u - b(t)\, e^{-A(t)},$$
$$g'(t) = -b(t)\, e^{-A(t)}.$$

A solution for function g is then given by

$$g(t) = -\int b(t)\, e^{-A(t)}\, dt.$$

Having that function g, we get a potential function

$$\psi(t,u) = e^{-A(t)} u - \int b(t)\, e^{-A(t)}\, dt.$$

All solutions y to the linear differential equation in (1.4.18) satisfy the equation

$$e^{-A(t)} y(t) - \int b(t)\, e^{-A(t)}\, dt = c_0,$$

where $c_0 \in \mathbb{R}$ is arbitrary. This is the implicit form of the solution. In this case it is simple to find the explicit formof the solution, which is given by

$$y(t) = e^{A(t)} \Big[c_0 + \int b(t)\, e^{-A(t)}\, dt \Big].$$

This expression agrees with the one in Theorem 1.2.4, when we studied linear equations. ◁

1.4.5. **The Integrating Factor for the Inverse Function.** If a differential equation for a function y is exact, then the equation for the inverse function y^{-1} is also exact. When the equation for y is not exact, Theorem 1.4.4 says when there is an integrating factor and how to find it. Sometimes the integrating factor for the differential equation for y does not exist, but the integrating factor for the differential equation for y^{-1} does, infact, exist. We study this situation in a bit more detail now.

Let us use the notation $y(x)$ for the function values, and $x(y)$ for the inverse function values. This is common in the literature. So in this last part of the section we replace the variable t by x.

Theorem 1.4.5. *If a differential equation is exact, as defined in this section, and a solution is invertible, then the differential equation for the inverse function is also exact.*

Proof of Theorem 1.4.5: Write the differential equation of a function y with values $y(x)$,

$$N(x,y)\, y' + M(x,y) = 0.$$

We have assumed that the equation is exact, so in this notation $\partial_x N = \partial_y M$. If a solution y is invertible and we use the notation $y^{-1}(y) = x(y)$, we have the well-known relation

$$x'(y) = \frac{1}{y'(x(y))}.$$

Divide the differential equation above by y' and use the relation above, then we get

$$N(x,y) + M(x,y)\, x' = 0,$$

where now y is the independent variable and the unknwon function is x, with values $x(y)$, and the prime means $x' = dx/dy$. The condition for this last equation to be exact is

$$\partial_y M = \partial_x N,$$

which we know holds because the equation for y is exact. This establishes the Theorem. □

Suppose now that a differential equation $N(x,y)\, y' + M(x,y) = 0$ is not exact. In Theorem 1.4.4 we have seen that a non-exact differential equation can be transformed into an exact equation in the case that the function $(\partial_y M - \partial_x N)/N$ does not depend on y. In that case the differential equation

$$(\mu\, N)\, y' + (\mu M) = 0$$

is exact when the integrating factor function μ is a solution of the equation

$$\frac{\mu'(x)}{\mu(x)} = \frac{(\partial_y M - \partial_x N)}{N}.$$

If the function $(\partial_y M - \partial_x N)/N$ does depend on y, the integrating factor μ for the equation for y does not exist. But the integrating factor for the equation for the inverse function x might exist. The following result says when this is the case.

Theorem 1.4.6 (Integrating factor II). *Assume that the differential equation*

$$M(x, y)\, x' + N(x, y) = 0 \tag{1.4.22}$$

is not exact because $\partial_y M(x, y) \neq \partial_x N(x, y)$ holds for the continuously differentiable functions M, N on their domain $R = (x_1, x_2) \times (y_1, y_2)$. If $M \neq 0$ on R and the function

$$-\frac{1}{M(x, y)} \big[\partial_y M(x, y) - \partial_x N(x, y)\big] \tag{1.4.23}$$

does not depend on the variable y, then the equation below is exact,

$$(\mu\, M)\, x' + (\mu\, N) = 0 \tag{1.4.24}$$

where the function μ, which depends only on $y \in (y_1, y_2)$, is a solution of the equation

$$\frac{\mu'(y)}{\mu(y)} = -\frac{1}{M(x, y)} \big[\partial_y M(x, y) - \partial_x N(x, y)\big].$$

Proof of Theorem 1.4.6: We know that the original differential equation in (1.4.22) is not exact because $\partial_y M \neq \partial_x N$. Now multiply the differential equation by a non-zero function μ that depends only on y,

$$(\mu M)\, x' + (\mu N) = 0. \tag{1.4.25}$$

We look for a function μ such that this new equation is exact. This means that μ must satisfy the equation

$$\partial_y(\mu M) = \partial_x(\mu N).$$

Recalling that μ depends only on y and denoting $\partial_y \mu = \mu'$, we get

$$\mu' M + \mu\, \partial_y M = \mu\, \partial_x N \quad \Rightarrow \quad \mu' M = \mu\, (\partial_x N - \partial_y M).$$

So the differential equation in (1.4.13) is exact iff holds

$$\frac{\mu'}{\mu} = -\frac{(\partial_y M - \partial_x N)}{M},$$

and a necessary condition for such an equation to have solutions is that the right-hand side be independent of the variable x. This establishes the Theorem. □

1.4.6. Exercises.

1.4.1.- Consider the equation
$$(1+t^2)\,y' = -2t\,y.$$
(a) Determine whether the differential equation is exact.
(b) Find every solution of the equation above.

1.4.2.- Consider the equation
$$t\cos(y)\,y' - 2y\,y' = -t - \sin(y).$$
(a) Determine whether the differential equation is exact.
(b) Find every solution of the equation above.

1.4.3.- Consider the equation
$$y' = \frac{-2 - y\,e^{ty}}{-2y + t\,e^{ty}}.$$
(a) Determine whether the differential equation is exact.
(b) Find every solution of the equation above.

1.4.4.- Consider the equation
$$(6x^5 - xy) + (-x^2 + xy^2)y' = 0,$$
with initial condition $y(0) = 1$.
(a) Find the integrating factor μ that converts the equation above into an exact equation.
(b) Find an implicit expression for the solution y of the IVP.

1.4.5.- Consider the equation
$$\Big(2x^2y + \frac{y}{x^2}\Big)\,y' + 4xy^2 = 0,$$
with initial condition $y(0) = -2$.
(a) Find the integrating factor μ that converts the equation above into an exact equation.
(b) Find an implicit expression for the solution y of the IVP.

1.5. Applications

Different physical systems may be described by the same mathematical structure. The radioactive decay of a substance, the cooling of a solid material, or the salt concentration on a water tank can be described with linear differential equations. A radioactive substance decays at a rate proprotional to the substance amount at the time. Something similar happens to the temperature of a cooling body. Linear, constant coefficients, differential equations describe these two situations. The salt concentration inside a water tank changes in the case that salty water is allowed in and out of the tank. This situation is described with a linear variable coefficients differential equation.

1.5.1. **Radioactive Decay.**

Radioactive decay occurs in the nuclei of certain substances, for example Uranium-235, Radium-226, Radon-222, Polonium-218, Lead-214, Cobalt-60, Carbon-14, etc. The nuclei emit different types of particles and end up in states of lower energy. There are many types of radioactive decays. Certain nuclei emit alpha particles (Helium nuclei), other nuclei emit protons (Hydrogen nuclei), even other nuclei emit an electrons, gamma-rays, neutrinos, etc. The radioactive decay of a single nucleus cannot be predicted but the decay of a large number can. The rate of change in the amount of a radioactive substance in a sample is proportional to the negative of that amount.

Definition 1.5.1. *The amount N of a radioactive substance in a sample as function of time satifies the* ***radioactive decay equation*** *iff the function N is solution of*

$$N'(t) = -k\, N(t),$$

where $k > 0$ is called the decay constant and characterizes the radioactive material.

The differential equation above is both linear and separable. We choose to solve it using the integrating factor method. The integrating factor is e^{kt}, then

$$\big[N'(t) + k\, N(t)\big] e^{kt} = 0 \quad \Leftrightarrow \quad \big[e^{kt} N(t)\big]' = 0.$$

We get that all solutions of the radioactive decay equations are given by

$$N(t) = N_0\, e^{-kt},$$

where $N_0 = N(0)$ is the initial amount of the substance. The amount of a radioactive material in a sample decays exponentially in time.

Remark: Radioactive materials are often characterized not by their decay constant k but by their half-life τ. This is a time such that half of the original amout of the radioactive substance has decayed.

Definition 1.5.2. *The* ***half-life*** *of a radioactive substance is the time τ such that*

$$N(\tau) = \frac{N_0}{2},$$

where N_0 is the initial amount of the radioactive substance.

There is a simple relation between the material constant and the material half-life.

Theorem 1.5.3. *A radioactive material constant k and half-life τ are related by the equation*

$$k\tau = \ln(2).$$

Proof of Theorem 1.5.3: We know that the amount of a radioactive material as function of time is given by

$$N(t) = N_0\, e^{-kt}.$$

Then, the definition of half-life implies,

$$\frac{N_0}{2} = N_0\, e^{-k\tau} \quad \Rightarrow \quad -k\tau = \ln\Big(\frac{1}{2}\Big) \quad \Rightarrow \quad k\tau = \ln(2).$$

This establishes the Theorem. □

The amount of a radioactive material, N, can be expressed in terms of the half-life,

$$N(t) = N_0\, e^{(-t/\tau)\ln(2)} \quad \Rightarrow \quad N(t) = N_0\, e^{\ln[2^{(-t/\tau)}]} \quad \Rightarrow \quad N(t) = N_0\, 2^{-t/\tau}.$$

From this last expression is clear that for $t = \tau$ we get $N(\tau) = N_0/2$.

Our first example is about dating remains with Carbon-14. The Carbon-14 is a radioactive isotope of Carbon-12 with a half-life of $\tau = 5730$ years. Carbon-14 is being constantly created in the atmosphere and is accumulated by living organisms. While the organism lives, the amount of Carbon-14 in the organism is held constant. The decay of Carbon-14 is compensated with new amounts when the organism breaths or eats. When the organism dies, the amount of Carbon-14 in its remains decays. So the balance between normal and radioactive carbon in the remains changes in time.

Example 1.5.1: If certain remains are found containing an amount of 14 % of the original amount of Carbon-14, find the date of the remains.

Solution: Suppose that $t = 0$ is set at the time when the organism dies. If at the present time t the remains contain 14% of the original amount, that means

$$N(t) = \frac{14}{100}\, N_0.$$

Since Carbon-14 is a radioactive substant with half-life τ, the amount of Carbon-14 decays in time as follows,

$$N(t) = N_0\, 2^{-t/\tau},$$

where $\tau = 5730$ years is the Carbon-14 half-life. Therefore,

$$2^{-t/\tau} = \frac{14}{100} \quad \Rightarrow \quad -\frac{t}{\tau} = \log_2(14/100) \quad \Rightarrow \quad t = \tau \log_2(100/14).$$

We obtain that $t = 16,253$ years. The organism died more that $16,000$ years ago. ◁

1.5.2. **Newton's Cooling Law.** The Newton cooling law describes how objects cool down when they are placed in a medium held at a constant temperature.

Definition 1.5.4. *The* ***Newton cooling law*** *says that the temperature T at a time t of a material placed in a surrounding medium held at a constant temperature T_s satisfies*

$$(\Delta T)' = -k\,(\Delta T),$$

with $\Delta T(t) = T(t) - T_s$, and $k > 0$, constant, characterizing the material thermal properties.

Remark: The Newton cooling law equation for the temperature difference ΔT is the same as the radioactive decay equation.

We know that the solution is for the temperatre difference is $(\Delta T)(t) = (\Delta T)_0\, e^{-kt}$, where $(\Delta T)_0 = T(0) - T_s$. So we get,

$$(T - T_s)(t) = (T_0 - T_s)\, e^{-kt} \quad \Rightarrow \quad T(t) = (T_0 - T_s)\, e^{-kt} + T_s.$$

where the constant $k > 0$ depends on the material and the surrounding medium.

EXAMPLE 1.5.2: A cup with water at 45 C is placed in the cooler held at 5 C. If after 2 minutes the water temperature is 25 C, when will the water temperature be 15 C?

SOLUTION: We know that the solution of the Newton cooling law equation is

$$T(t) = (T_0 - T_s)\, e^{-kt} + T_s,$$

and we also know that in this case we have

$$T_0 = 45, \qquad T_s = 5, \qquad T(2) = 25.$$

In this example we need to find t_1 such that $T(t_1) = 15$. In order to find that t_1 we first need to find the constant k,

$$T(t) = (45 - 5)\, e^{-kt} + 5 \quad \Rightarrow \quad T(t) = 40\, e^{-kt} + 5.$$

Now use the fact that $T(2) = 25$ C, that is,

$$20 = T(2) = 40\, e^{-2k} \quad \Rightarrow \quad \ln(1/2) = -2k \quad \Rightarrow \quad k = \frac{1}{2}\ln(2).$$

Having the constant k we can now go on and find the time t_1 such that $T(t_1) = 15$ C.

$$T(t) = 40\, e^{-t\ln(\sqrt{2})} + 5 \quad \Rightarrow \quad 10 = 40\, e^{-t_1 \ln(\sqrt{2})} \quad \Rightarrow \quad t_1 = 4.$$

◁

1.5.3. **Salt in a Water Tank.** We study the system pictured in Fig. 3. A tank has a salt mass $Q(t)$ dissolved in a volume $V(t)$ of water at a time t. Water is pouring into the tank at a rate $r_i(t)$ with a salt concentration $q_i(t)$. Water is also leaving the tank at a rate $r_o(t)$ with a salt concentration $q_o(t)$. Recall that a water rate r means water volume per unit time, and a salt concentration q means salt mass per unit volume.

We assume that the salt entering in the tank gets instantaneously mixed. As a consequence the salt concentration in the tank is homogeneous at every time. This property simplifies the mathematical model describing the salt in the tank.

Before stating the problem we want to solve, we review the physical units of the main fields involved in it. Denote by $[r_i]$ the units of the quantity r_i. Then we have

$$[r_i] = [r_o] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i] = [q_o] = \frac{\text{Mass}}{\text{Volume}},$$

$$[V] = \text{Volume}, \quad [Q] = \text{Mass}.$$

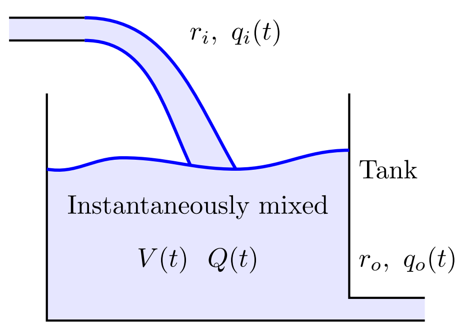


FIGURE 3. Description of the water tank problem.

Definition 1.5.5. *The* ***Water Tank Problem*** *refers to water coming into a tank at a rate r_i with salt concentration q_i, and going out the tank at a rate r_o and salt concentration q_o, so that the water volume V and the total amount of salt Q, which is instantaneously mixed, in the tank satisfy the following equations,*

$$V'(t) = r_i(t) - r_o(t), \tag{1.5.1}$$

$$Q'(t) = r_i(t)\, q_i(t) - r_o(t), q_o(t), \tag{1.5.2}$$

$$q_o(t) = \frac{Q(t)}{V(t)}, \tag{1.5.3}$$

$$r_i'(t) = r_o'(t) = 0. \tag{1.5.4}$$

The first and second equations above are just the mass conservation of water and salt, respectively. Water volume and mass are proportional, so both are conserved, and we chose the volume to write down this conservation in Eq. (1.5.1). This equation is indeed a conservation because it says that the water volume variation in time is equal to the difference of volume time rates coming in and going out of the tank. Eq. (1.5.2) is the salt mass conservation, since the salt mass variation in time is equal to the difference of the salt mass time rates coming in and going out of the tank. The product of a rate r times a concentration q has units of mass per time and represents the amount of salt entering or leaving the tank per unit time. Eq.(1.5.3) is implied by the instantaneous mixing mechanism in the tank. Since the salt is mixed instantaneously in the tank, the salt concentration in the tank is homogeneous with value $Q(t)/V(t)$. Finally the equations in (1.5.4) say that both rates in and out are time independent, that is, constants.

Theorem 1.5.6. *The amount of salt Q in a water tank problem defined in Def. 1.5.5 satisfies the differential equation*

$$Q'(t) = a(t)\,Q(t) + b(t), \tag{1.5.5}$$

where the coefficients in the equation are given by

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}, \qquad b(t) = r_i\,q_i(t). \tag{1.5.6}$$

Proof of Theorem 1.5.6: The equation for the salt in the tank given in (1.5.5) comes from Eqs. (1.5.1)-(1.5.4). We start noting that Eq. (1.5.4) says that the water rates are constant. We denote them as r_i and r_o. This information in Eq. (1.5.1) implies that V' is constant. Then we can easily integrate this equation to obtain

$$V(t) = (r_i - r_o)\,t + V_0, \tag{1.5.7}$$

where $V_0 = V(0)$ is the water volume in the tank at the initial time $t = 0$. On the other hand, Eqs.(1.5.2) and (1.5.3) imply that

$$Q'(t) = r_i\,q_i(t) - \frac{r_o}{V(t)}\,Q(t).$$

Since $V(t)$ is known from Eq. (1.5.7), we get that the function Q must be solution of the differential equation

$$Q'(t) = r_i\,q_i(t) - \frac{r_o}{(r_i - r_o)\,t + V_0}\,Q(t).$$

This is a linear ODE for the function Q. Indeed, introducing the functions

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}, \qquad b(t) = r_i\,q_i(t),$$

the differential equation for Q has the form

$$Q'(t) = a(t)\,Q(t) + b(t).$$

This establishes the Theorem. □

We could use the formula for the general solution of a linear equation given in Section 1.2 to write the solution of Eq. (1.5.5) for Q. Such formula covers all cases we are going to study in this section. Since we already know that formula, we choose to find solutions in particular cases. These cases are given by specific choices of the rate constants r_i, r_o, the concentration function q_i, and the initial data constants V_0 and $Q_0 = Q(0)$. The study of solutions to Eq. (1.5.5) in several particular cases might provide a deeper understanding of the physical situation under study than the expression of the solution Q in the general case.

Example 1.5.3: Consider a water tank problem with equal constant water rates $r_i = r_o = r$, with constant incoming concentration q_i, and with a given initial water volume in the tank V_0. Then, find the solution to the initial value problem

$$Q'(t) = a(t)\,Q(t) + b(t), \qquad Q(0) = Q_0,$$

where function a and b are given in Eq. (1.5.6). Graph the solution function Q for different values of the initial condition Q_0.

Solution: The assumption $r_i = r_o = r$ implies that the function a is constant, while the assumption that q_i is constant implies that the function b is also constant too,

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = a_0,$$
$$b(t) = r_i\,q_i(t) \quad \Rightarrow \quad b(t) = r_i\,q_i = b_0.$$

Then, we must solve the initial value problem for a constant coefficients linear equation,

$$Q'(t) = a_0\,Q(t) + b_0, \qquad Q(0) = Q_0,$$

The integrating factor method can be used to find the solution of the initial value problem above. The formula for the solution is given in Theorem 1.1.5,

$$Q(t) = \Big(Q_0 + \frac{b_0}{a_0}\Big)\,e^{a_0 t} - \frac{b_0}{a_0}.$$

In our case the we can evaluate the constant b_0/a_0, and the result is

$$\frac{b_0}{a_0} = (rq_i)\Big(-\frac{V_0}{r}\Big) \quad \Rightarrow \quad -\frac{b_0}{a_0} = q_i V_0.$$

Then, the solution Q has the form,

$$Q(t) = \big(Q_0 - q_i V_0\big)\,e^{-rt/V_0} + q_i V_0. \tag{1.5.8}$$

The initial amount of salt Q_0 in the tank can be any non-negative real number. The solution behaves differently for different values of Q_0. We classify these values in three classes:

(a) The initial amount of salt in the tank is the critical value $Q_0 = q_i V_0$. In this case the solution Q remains constant equal to this critical value, that is, $Q(t) = q_i V_0$.
(b) The initial amount of salt in the tank is bigger than the critical value, $Q_0 > q_i V_0$. In this case the salt in the tank Q decreases exponentially towards the critical value.
(c) The initial amount of salt in the tank is smaller than the critical value, $Q_0 < q_i V_0$. In this case the salt in the tank Q increases exponentially towards the critical value.

The graphs of a few solutions in these three classes are plotted in Fig. 4.

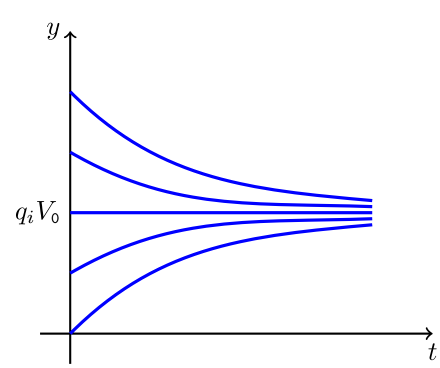


Figure 4. The function Q in (1.5.8) for a few values of the initial condition Q_0.

◁

Example 1.5.4: Consider a water tank problem with equal constant water rates $r_i = r_o = r$ and fresh water is coming into the tank, hence $q_i = 0$. Then, find the time t_1 such that the salt concentration in the tank $Q(t)/V(t)$ is 1% the initial value. Write that time t_1 in terms of the rate r and initial water volume V_0.

Solution: The first step to solve this problem is to find the solution Q of the initial value problem

$$Q'(t) = a(t)\, Q(t) + b(t), \qquad Q(0) = Q_0,$$

where function a and b are given in Eq. (1.5.6). In this case they are

$$a(t) = -\frac{r_o}{(r_i - r_o)\, t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0},$$
$$b(t) = r_i\, q_i(t) \quad \Rightarrow \quad b(t) = 0.$$

The initial value problem we need to solve is

$$Q'(t) = -\frac{r}{V_0}\, Q(t), \qquad Q(0) = Q_0.$$

From Section 1.1 we know that the solution is given by

$$Q(t) = Q_0\, e^{-rt/V_0}.$$

We can now proceed to find the time t_1. We first need to find the concentration $Q(t)/V(t)$. We already have $Q(t)$ and we now that $V(t) = V_0$, since $r_i = r_o$. Therefore,

$$\frac{Q(t)}{V(t)} = \frac{Q(t)}{V_0} = \frac{Q_0}{V_0}\, e^{-rt/V_0}.$$

The condition that defines t_1 is

$$\frac{Q(t_1)}{V(t_1)} = \frac{1}{100}\, \frac{Q_0}{V_0}.$$

From these two equations above we conclude that

$$\frac{1}{100}\, \frac{Q_0}{V_0} = \frac{Q(t_1)}{V(t_1)} = \frac{Q_0}{V_0}\, e^{-rt_1/V_0}.$$

The time t_1 comes from the equation

$$\frac{1}{100} = e^{-rt_1/V_0} \quad \Leftrightarrow \quad \ln\Big(\frac{1}{100}\Big) = -\frac{rt_1}{V_0} \quad \Leftrightarrow \quad \ln(100) = \frac{rt_1}{V_0}.$$

The final result is given by

$$t_1 = \frac{V_0}{r}\, \ln(100).$$

◁

Example 1.5.5: Consider a water tank problem with equal constant water rates $r_i = r_o = r$, with only fresh water in the tank at the initial time, hence $Q_0 = 0$ and with a given initial volume of water in the tank V_0. Then find the function salt in the tank Q if the incoming salt concentration is given by the function

$$q_i(t) = 2 + \sin(2t).$$

Solution: We need to find the solution Q to the initial value problem

$$Q'(t) = a(t)\, Q(t) + b(t), \qquad Q(0) = 0,$$

where function a and b are given in Eq. (1.5.6). In this case we have

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0} \quad \Rightarrow \quad a(t) = -\frac{r}{V_0} = -a_0,$$

$$b(t) = r_i\, q_i(t) \quad \Rightarrow \quad b(t) = r\,\big[2 + \sin(2t)\big].$$

We are changing the sign convention for a_0 so that $a_0 > 0$. The initial value problem we need to solve is

$$Q'(t) = -a_0\, Q(t) + b(t), \qquad Q(0) = 0.$$

The solution is computed using the integrating factor method and the result is

$$Q(t) = e^{-a_0 t} \int_0^t e^{a_0 s} b(s)\, ds,$$

where we used that the initial condition is $Q_0 = 0$. Recalling the definition of the function b we obtain

$$Q(t) = e^{-a_0 t} \int_0^t e^{a_0 s} \big[2 + \sin(2s)\big]\, ds.$$

This is the formula for the solution of the problem, we only need to compute the integral given in the equation above. This is not straightforward though. We start with the following integral found in an integration table,

$$\int e^{ks} \sin(ls)\, ds = \frac{e^{ks}}{k^2 + l^2}\,\big[k \sin(ls) - l\cos(ls)\big],$$

where k and l are constants. Therefore,

$$\int_0^t e^{a_0 s}\big[2 + \sin(2s)\big]\, ds = \Big[\frac{2}{a_0} e^{a_0 s}\Big]\Big|_0^t + \Big[\frac{e^{a_0 s}}{a_0^2 + 2^2}\,\big[a_0 \sin(2s) - 2\cos(2s)\big]\Big]\Big|_0^t,$$

$$= \frac{2}{a_0} q\big(e^{a_0 t} - 1\big) + \frac{e^{a_0 t}}{a_0^2 + 2^2}\,\big[a_0 \sin(2t) - 2\cos(2t)\big] + \frac{2}{a_0^2 + 2^2}.$$

With the integral above we can compute the solution Q as follows,

$$Q(t) = e^{-a_0 t}\Big[\frac{2}{a_0}\big(e^{a_0 t} - 1\big) + \frac{e^{a_0 t}}{a_0^2 + 2^2}\,\big[a_0 \sin(2t) - 2\cos(2t)\big] + \frac{2}{a_0^2 + 2^2}\Big],$$

recalling that $a_0 = r/V_0$. We rewrite expression above as follows,

$$Q(t) = \frac{2}{a_0} + \Big[\frac{2}{a_0^2 + 2^2} - \frac{2}{a_0}\Big]\, e^{-a_0 t} + \frac{1}{a_0^2 + 2^2}\,\big[a_0 \sin(2t) - 2\cos(2t)\big]. \tag{1.5.9}$$

$\triangleleft$

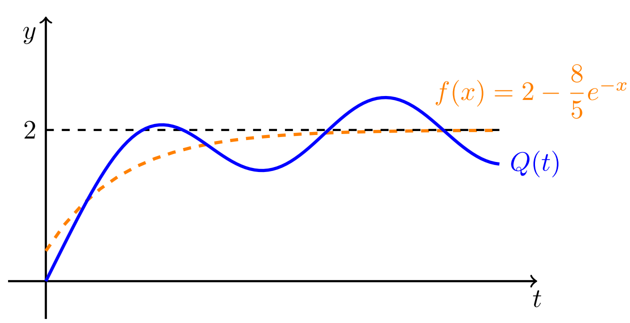


FIGURE 5. The graph of the function Q given in Eq. (1.5.9) for $a_0 = 1$.

1.5.4. **Exercises.**

1.5.1.- A radioactive material decays at a rate proportional to the amount present. Initially there are 50 milligrams of the material present and after one hour the material has lost 80% of its original mass.

(a) Find the mass of the material as function of time.
(b) Find the mass of the material after four hours.
(c) Find the half-life of the material.

1.5.2.- A tank initially contains $V_0 = 100$ liters of water with $Q_0 = 25$ grams of salt. The tank is rinsed with fresh water flowing in at a rate of $r_i = 5$ liters per minute and leaving the tank at the same rate. The water in the tank is well-stirred. Find the time such that the amount the salt in the tank is $Q_1 = 5$ grams.

1.5.3.- A tank initially contains $V_0 = 100$ liters of pure water. Water enters the tank at a rate of $r_i = 2$ liters per minute with a salt concentration of $q_1 = 3$ grams per liter. The instantaneously mixed mixture leaves the tank at the same rate it enters the tank. Find the salt concentration in the tank at any time $t \geqslant 0$. Also find the limiting amount of salt in the tank in the limit $t \to \infty$.

1.5.4.- A tank with a capacity of $V_m = 500$ liters originally contains $V_0 = 200$ liters of water with $Q_0 = 100$ grams of salt in solution. Water containing salt with concentration of $q_i = 1$ gram per liter is poured in at a rate of $r_i = 3$ liters per minute. The well-stirred water is allowed to pour out the tank at a rate of $r_o = 2$ liters per minute. Find the salt concentration in the tank at the time when the tank is about to overflow. Compare this concentration with the limiting concentration at infinity time if the tank had infinity capacity.

1.6. Nonlinear Equations

Linear differential equations are simpler to solve than nonlinear differential equations. We have found an explicit formula for the solutions to all linear equations, given in Theorem 1.2.4, while no general formula exists for all nonlinear equations. In §§ 1.2-1.4 we solved different types of nonlinear equations using different methods and arrived at different formulas for their solutions. And the nonlinear equations we solved are just a tiny part of all possible nonlinear equations.

We start this section with the Picard-Lindelöf Theorem. This statement says that a large class of nonlinear differential equations have solutions. But it does not provide a formula for the solutions. The proof of the Picard-Lindelöf Theorem is important. This proof provides an iteration to construct approximations to the solutions of differential equations. We end this section highlighting the main differences between solutions to linear and nonlinear differential equations.

1.6.1. The Picard-Lindelöf Theorem.

There exists no formula for the solutions of all nonlinear differential equations. We will prove this statement in the second part of this section. So there is no point in looking for such formula. What we can do, however, is to show whether a nonlinear differential equations have solutions or not. And whether the solution of an initial value problem is unique or not. This information is certainly less than having a formula for the solutions. But it is nonetheless valuable information. Results like this one are called existence and uniqueness statements about solutions to nonlinear differential equations.

We start with a precise definition of the nonlinear differential we are going to study.

Definition 1.6.1. *An ordinary differential equation $y'(t) = f(t, y(t))$ is called **nonlinear** iff the function f is nonlinear in the second argument.*

Example 1.6.1:

(a) The differential equation
$$y'(t) = \frac{t^2}{y^3(t)}$$
is nonlinear, since the function $f(t,u) = t^2/u^3$ is nonlinear in the second argument.

(b) The differential equation
$$y'(t) = 2ty(t) + \ln\big(y(t)\big)$$
is nonlinear, since the function $f(t,u) = 2tu + \ln(u)$ is nonlinear in the second argument, due to the term $\ln(u)$.

(c) The differential equation
$$\frac{y'(t)}{y(t)} = 2t^2$$
is linear, since the function $f(t,u) = 2t^2 u$ is linear in the second argument.

◁

The main result for this section is the Picard-Lindelöf Theorem. This is an existence and uniqueness result. It states what type of nonlinear differential equations have solutions. It also states when the solution of an initial value problem is unique or not. This statement does not provide a formula for the solutions. The proof is to construct a sequence of approximate solution of the differential equation and to show that this sequence coverges to a unique limit. That limit is the solution of the initial value problem for the nonlinear differential equation.

Theorem 1.6.2 (Picard-Lindelöf). *Consider the initial value problem*

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0. \tag{1.6.1}$$

If $f : S \to \mathbb{R}$ is continuous on the square $S = [t_0 - a, t_0 + a] \times [y_0 - a, y_0 + a] \subset \mathbb{R}^2$, for some $a > 0$, and satisfies the Lipschitz condition that there exists $k > 0$ such that

$$|f(t, y_2) - f(t, y_1)| < k\,|y_2 - y_1|,$$

for all (t, y_2), $(t, y_1) \in S$, then there exists a positive $b < a$ such that there exists a unique solution $y : [t_0 - b, t_0 + b] \to \mathbb{R}$ to the initial value problem in (1.6.1).

Remark: For the proof we start rewriting the differential equation as an integral equation for the unknown function y. We use this integral equation to construct a sequence of approximate solutions $\{y_n\}$ to the original initial value problem. We show that this sequence of approximate solutions has a unique limit as $n \to \infty$. We end the proof showing that this limit is the solution of the original initial value problem.

Remark: The proof below follows [15] § 1.6 and Zeidler's [16] § 1.8. It is important to read the review on complete normed vector spaces, called Banach spaces, given in these references.

Proof of Theorem 1.6.2: We must write the differential equation in 1.6.1 as an integral equation. So, integrate on both sides of that equation with respect to t,

$$\int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} f(s, y(s))\,ds \quad \Rightarrow \quad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds. \tag{1.6.2}$$

We have used the Fundamental Theorem of Calculus on the left-hand side of the first equation to get the second equation. And we have introduced the initial condition $y(t_0) = y_0$. We use this integral form of the original differential equation to construct a sequence of functions $\{y_n\}_{n=0}^{\infty}$. The domain of every function in this sequence is $D_a = [t_0 - a, t_0 + a]$. The sequence is defined as follows,

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds, \qquad n \geqslant 0, \qquad y_0(t) = y_0. \tag{1.6.3}$$

We see that the first element in the sequence is the constant function determined by the initial conditions in (1.6.1). The iteration in (1.6.3) is called the Picard iteration. The central idea of the proof is to show that the sequence $\{y_n\}$ is a Cauchy sequence in the space $C(D_b)$ of uniformly continuous functions in the domain $D_b = [t_0 - b, t_0 + b]$ for a small enough $b > 0$. This function space is a Banach space under the norm

$$\|u\| = \max_{t \in D_b} |u(t)|.$$

See [15] and references therein for the definition of Banach spaces and the proof that $C(D_b)$ with that norm is a Banach space. We now show that the sequence $\{y_n\}$ is a Cauchy sequence in that space. Any two consecutive elements in the sequence satisfy

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \max_{t \in D_b} \Big| \int_{t_0}^{t} f(s, y_n(s))\,ds - \int_{t_0}^{t} f(s, y_{n-1}(s))\,ds \Big| \\
&\leqslant \max_{t \in D_b} \int_{t_0}^{t} \big| f(s, y_n(s)) - f(s, y_{n-1}(s)) \big|\,ds \\
&\leqslant k \max_{t \in D_b} \int_{t_0}^{t} |y_n(s) - y_{n-1}(s)|\,ds \\
&\leqslant kb\,\|y_n - y_{n-1}\|.
\end{aligned}$$

Denoting $r = kb$, we have obtained the inequality

$$\|y_{n+1} - y_n\| \leqslant r\,\|y_n - y_{n-1}\| \quad \Rightarrow \quad \|y_{n+1} - y_n\| \leqslant r^n\,\|y_1 - y_0\|.$$

Using the triangle inequality for norms and and the sum of a geometric series one compute the following,

$$\begin{aligned}
\|y_n - y_{n+m}\| &= \|y_n - y_{n+1} + y_{n+1} - y_{n+2} + \cdots + y_{n+(m-1)} - y_{n+m}\| \\
&\leqslant \|y_n - y_{n+1}\| + \|y_{n+1} - y_{n+2}\| + \cdots + \|y_{n+(m-1)} - y_{n+m}\| \\
&\leqslant (r^n + r^{n+1} + \cdots + r^{n+m})\,\|y_1 - y_0\| \\
&\leqslant r^n(1 + r + r^2 + \cdots + r^m)\,\|y_1 - y_0\| \\
&\leqslant r^n\Big(\frac{1 - r^m}{1 - r}\Big)\,\|y_1 - y_0\|.
\end{aligned}$$

Now choose the positive constant b such that $b < \min\{a, 1/k\}$, hence $0 < r < 1$. In this case the sequence $\{y_n\}$ is a Cauchy sequence in the Banach space $C(D_b)$, with norm $\|\ \|$, hence converges. Denote the limit by $y = \lim_{n\to\infty} y_n$. This function satisfies the equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds,$$

which says that y is not only continuous but also differentiable in the interior of D_b, hence y is solution of the initial value problem in (1.6.1). The proof of uniqueness is left as an exercise. This establishes the Theorem. □

Example 1.6.2: Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = t\,y, \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t s\,y(s)\,ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t s\,y(s)\,ds.$$

Using the initial condition, $y(0) = 1$, we get the integral equation

$$y(t) = 1 + \int_0^t s\,y(s)\,ds.$$

We now define the sequence of approximate solutions, $\{y_n\}_{n=0}^{\infty}$, as follows,

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t s\,y_n(s)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence. We start computing y_1,

$$n = 0, \quad y_1(t) = 1 + \int_0^t s\,y_0(s)\,ds = 1 + \int_0^t s\,ds = 1 + \frac{t^2}{2}.$$

So $y_0 = 1$, and $y_1 = 1 + \dfrac{t^2}{2}$. Let's compute y_2,

$$y_2 = 1 + \int_0^t s\,y_1(s)\,ds = 1 + \int_0^t \Big(s + \frac{s^3}{2}\Big)\,ds = 1 + \frac{t^2}{2} + \frac{t^4}{8}.$$

So we've got $y_2(t) = 1 + \Big(\dfrac{t^2}{2}\Big) + \dfrac{1}{2}\Big(\dfrac{t^2}{2}\Big)^2$. In the same way it can be computed y_3, which is left as an exercise. The result is

$$y_3(t) = 1 + \Big(\frac{t^2}{2}\Big) + \frac{1}{2!}\Big(\frac{t^2}{2}\Big)^2 + \frac{1}{3!}\Big(\frac{t^2}{2}\Big)^3.$$

By computing few more terms one finds

$$y_n(t) = \sum_{k=0}^{n} \frac{1}{k!}\Big(\frac{t^2}{2}\Big)^k.$$

Hence the limit $n \to \infty$ is given by

$$y(t) = \lim_{n\to\infty} y_n(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\frac{t^2}{2}\Big)^k = e^{t^2/2}.$$

The last equality above follows from the expansion $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$, with $x = t^2/2$. So we conclude, $y(t) = e^{t^2/2}$. ◁

Remark: The differential equation $y' = t\,y$ is of course separable, so the solution to the initial value problem in Example 1.6.2 can be obtained using the methods in Section 1.3,

$$\frac{y'}{y} = t \quad \Rightarrow \quad \ln(y) = \frac{t^2}{2} + c \quad \Rightarrow \quad y(t) = \tilde{c}\, e^{t^2/2}; \quad 1 = y(0) = \tilde{c} \quad \Rightarrow \quad y(t) = e^{t^2/2}.$$

Example 1.6.3: Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t (2\,y(s) + 3)\,ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t (2\,y(s) + 3)\,ds.$$

Using the initial condition, $y(0) = 1$,

$$y(t) = 1 + \int_0^t (2\,y(s) + 3)\,ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = 1, \quad y_{n+1}(t) = 1 + \int_0^t (2\,y_n(s) + 3)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence. We said $y_0 = 1$, now y_1 is given by

$$n = 0, \quad y_1(t) = 1 + \int_0^t (2\,y_0(s) + 3)\,ds = 1 + \int_0^t 5\,ds = 1 + 5t.$$

So $y_1 = 1 + 5t$. Now we compute y_2,

$$y_2 = 1+\int_0^t (2\,y_1(s)+3)\,ds = 1+\int_0^t \big(2(1+5s)+3\big)\,ds \quad \Rightarrow \quad y_2 = 1+\int_0^t \big(5+10s\big)\,ds = 1+5t+5t^2.$$

So we've got $y_2(t) = 1 + 5t + 5t^2$. Now y_3,

$$y_3 = 1 + \int_0^t (2\,y_2(s) + 3)\,ds = 1 + \int_0^t \big(2(1 + 5s + 5s^2) + 3\big)\,ds$$

so we have,

$$y_3 = 1 + \int_0^t \big(5 + 10s + 10s^2\big)\,ds = 1 + 5t + 5t^2 + \frac{10}{3}\,t^3.$$

So we obtained $y_3(t) = 1 + 5t + 5t^2 + \frac{10}{3}\,t^3$. We now try reorder terms in this last expression so we can get a power series expansion we can write in terms of simple functions. The first step is identify common factors, like the factor five in y_3,

$$y_3(t) = 1 + 5\,\Big(t + t^2 + \frac{2}{3}\,t^3\Big).$$

We now try to rewrite the expression above to get an $n!$ in the denominator of each term with a power t^n, that is,

$$y_3(t) = 1 + 5\,\Big(t + \frac{2t^2}{2!} + \frac{4t^3}{3!}\Big).$$

We then realize that we can rewrite the expression above in terms of power of $(2t)$, that is,

$$y_3(t) = 1 + 5\,\frac{2}{2}\,\Big(t + \frac{2t^2}{2!} + \frac{4t^3}{3!}\Big) = 1 + \frac{5}{2}\,\Big((2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\Big).$$

From this last expressionis simple to guess the n-th approximation

$$y_n(t) = 1 + \frac{5}{2}\,\Big((2t) + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots + \frac{(2t)^n}{n!}\Big) \quad\Rightarrow\quad y_n(t) = 1 + \frac{5}{2}\sum_{k=1}^{\infty}\frac{(2t)^k}{k!}.$$

Recall now that the power series expansion for the exponential

$$e^{at} = \sum_{k=0}^{\infty}\frac{(at)^k}{k!}.$$

Notice that the sum in the exponential starts at $k = 0$, while the sum in y_n starts at $k = 1$. Then, the limit $n \to \infty$ is given by

$$y(t) = \lim_{n\to\infty} y_n(t) = 1 + \frac{5}{2}\sum_{k=1}^{\infty}\frac{(2t)^k}{k!} = 1 + \frac{5}{2}\big(e^{2t} - 1\big),$$

We have been able to add the power series and we have the solution written in terms of simple functions. We have used the expansion for the exponential function

$$e^{at} - 1 = (at) + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots = \sum_{k=1}^{\infty}\frac{(at)^k}{k!}$$

with $a = 2$. One last rewriting of the solution and we obtain

$$y(t) = \frac{5}{2}e^{2t} - \frac{3}{2}.$$

◁

Remark: The differential equation $y' = 2\,y + 3$ is of course linear, so the solution to the initial value problem in Example 1.6.3 can be obtained using the methods in Section 1.1,

$$e^{-2t}\,(y' - 2\,y) = e^{-2t}\,3 \quad\Rightarrow\quad e^{-2t}\,y = -\frac{3}{2}\,e^{-2t} + c \quad\Rightarrow\quad y(t) = c\,e^{2t} - \frac{3}{2};$$

and the initial condition implies

$$1 = y(0) = c - \frac{3}{2} \quad\Rightarrow\quad c = \frac{5}{2} \quad\Rightarrow\quad y(t) = \frac{5}{2}\,e^{2t} - \frac{3}{2}.$$

Example 1.6.4: Use the proof of Picard-Lindelöf's Theorem to find the solution to

$$y' = a\,y + b \qquad y(0) = \hat{y}_0, \qquad a, b \in \mathbb{R}.$$

Solution: We first transform the differential equation into an integral equation.

$$\int_0^t y'(s)\,ds = \int_0^t (a\,y(s) + b)\,ds \quad \Rightarrow \quad y(t) - y(0) = \int_0^t (a\,y(s) + b)\,ds.$$

Using the initial condition, $y(0) = \hat{y}_0$,

$$y(t) = \hat{y}_0 + \int_0^t (a\,y(s) + b)\,ds.$$

We now define the sequence of approximate solutions:

$$y_0 = y(0) = \hat{y}_0, \quad y_{n+1}(t) = \hat{y}_0 + \int_0^t (a\,y_n(s) + b)\,ds, \quad n \geqslant 0.$$

We now compute the first elements in the sequence. We said $y_0 = \hat{y}_0$, now y_1 is given by

$$\begin{aligned} n = 0, \quad y_1(t) &= y_0 + \int_0^t (a\,y_0(s) + b)\,ds \\ &= \hat{y}_0 + \int_0^t (a\,\hat{y}_0 + b)\,ds \\ &= \hat{y}_0 + (a\,\hat{y}_0 + b)t. \end{aligned}$$

So $y_1 = \hat{y}_0 + (a\,\hat{y}_0 + b)t$. Now we compute y_2,

$$\begin{aligned} y_2 &= \hat{y}_0 + \int_0^t [a\,y_1(s) + b]\,ds \\ &= 1 + \int_0^t \big[a(\hat{y}_0 + (a\,\hat{y}_0 + b)s) + b\big]\,ds \\ &= \hat{y}_0 + (a\hat{y}_0 + b)t + (a\,\hat{y}_0 + b)\frac{at^2}{2} \end{aligned}$$

So we obtained $y_2(t) = \hat{y}_0 + (a\hat{y}_0 + b)t + (a\,\hat{y}_0 + b)\dfrac{at^2}{2}$. A similar calculation gives us y_3,

$$y_3(t) = \hat{y}_0 + (a\hat{y}_0 + b)t + (a\,\hat{y}_0 + b)\frac{at^2}{2} + (a\,\hat{y}_0 + b)\frac{a^2t^3}{3!}.$$

We now try reorder terms in this last expression so we can get a power series expansion we can write in terms of simple functions. The first step is identify common factors, like the factor $(a\,\hat{y}_0 + b)$ in y_3,

$$y_3(t) = \hat{y}_0 + (a\hat{y}_0 + b)\Big(t + \frac{at^2}{2} + \frac{a^2t^3}{3!}\Big).$$

We then realize that we can rewrite the expression above in terms of power of (at), that is,

$$\begin{aligned} y_3(t) &= \hat{y}_0 + (a\hat{y}_0 + b)\frac{a}{a}\Big(t + \frac{at^2}{2} + \frac{a^2t^3}{3!}\Big) \\ &= \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\Big((at) + \frac{(at)^2}{2} + \frac{(at)^3}{3!}\Big). \end{aligned}$$

From this last expressionis simple to guess the n-th approximation

$$y_n(t) = \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\Big((at) + \frac{(at)^2}{2} + \frac{(at)^3}{3!} + \cdots + \frac{(at)^n}{n!}\Big)$$
$$= \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\sum_{k=1}^{\infty}\frac{(at)^k}{k!}.$$

Recall now that the power series expansion for the exponential

$$e^{at} = \sum_{k=0}^{\infty}\frac{(at)^k}{k!}.$$

Notice that the sum in the exponential starts at $k = 0$, while the sum in y_n starts at $k = 1$. Then, the limit $n \to \infty$ is given by

$$y(t) = \lim_{n\to\infty} y_n(t)$$
$$= \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\sum_{k=1}^{\infty}\frac{(at)^k}{k!}$$
$$= \hat{y}_0 + \Big(\hat{y}_0 + \frac{b}{a}\Big)\big(e^{at} - 1\big),$$

We have been able to add the power series and we have the solution written in terms of simple functions. We have used the expansion for the exponential function

$$e^{at} - 1 = (at) + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots = \sum_{k=1}^{\infty}\frac{(at)^k}{k!}.$$

One last rewriting of the solution and we obtain

$$y(t) = \Big(\hat{y}_0 + \frac{b}{a}\Big)e^{at} - \frac{b}{a}.$$

$\lhd$

Remark: We reobtained Eq. (1.1.10) in Theorem 1.1.5.

1.6.2. **Comparison Linear Nonlinear Equations.** Let us recall the initial value problem for a linear differential equation. Given functions a, b and constants t_0, y_0, find a function y solution of the equations

$$y' = a(t)\,y + b(t), \qquad y(t_0) = y_0. \tag{1.6.4}$$

The main result regarding solutions to this problem is summarized in Theorem 1.2.4, which we reproduce it below.

Theorem 1.2.4 (Variable coefficients). *Given continuous functions $a, b : (t_1, t_2) \to \mathbb{R}$ and constants $t_0 \in (t_1, t_2)$ and $y_0 \in \mathbb{R}$, the initial value problem*

$$y' = a(t)\,y + b(t), \qquad y(t_0) = y_0, \tag{1.2.6}$$

has the unique solution $y : (t_1, t_2) \to \mathbb{R}$ given by

$$y(t) = e^{A(t)}\Big[y_0 + \int_{t_0}^{t} e^{-A(s)}\,b(s)\,ds\Big], \tag{1.2.7}$$

where we introduced the function $A(t) = \displaystyle\int_{t_0}^{t} a(s)\,ds$.

From the Theorem above we can see that solutions to linear differential equations satisfiy the following properties:

(a) There is an explicit expression for the solutions of a differential equations.
(b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution.
(c) For every initial condition $y_0 \in \mathbb{R}$ the solution $y(t)$ is defined for all (t_1, t_2).

Remark: None of these properties hold for solutions of nonlinear differential equations.

From the Picard-Lindelöf Theorem one can see that solutions to nonlinear differential equations satisfy the following properties:

(i) There is no explicit formula for the solution to every nonlinear differential equation.
(ii) Solutions to initial value problems for nonlinear equations may be non-unique when the function f does not satisfy the Lipschitz condition.
(iii) The domain of a solution y to a nonlinear initial value problem may change when we change the initial data y_0.

The next three examples (1.6.5)-(1.6.7) are particular cases of the statements in (i)-(iii). We start with an equation whose *solutions cannot be written in explicit form.* The reason is not lack of ingenuity, it has been proven that such explicit expression does not exist.

EXAMPLE 1.6.5: For every constant a_1, a_2, a_3, a_4, find all solutions y to the equation

$$y'(t) = \frac{t^2}{\big(y^4(t) + a_4\, y^3(t) + a_3\, y^2(t) + a_2\, y(t) + a_1\big)}. \tag{1.6.5}$$

SOLUTION: The nonlinear differential equation above is separable, so we follow § 1.3 to find its solutions. First we rewrite the equation as

$$\big(y^4(t) + a_4\, y^3(t) + a_3\, y^2(t) + a_2\, y(t) + a_1\big)\, y'(t) = t^2.$$

Then we integrate on both sides of the equation,

$$\int \big(y^4(t) + a_4\, y^3(t) + a_3\, y^2(t) + a_2\, y(t) + a_1\big)\, y'(t)\, dt = \int t^2\, dt + c.$$

Introduce the substitution $u = y(t)$, so $du = y'(t)\, dt$,

$$\int \big(u^4 + a_4\, u^3 + a_3\, u^2 + a_2\, u + a_1\big)\, du = \int t^2\, dt + c.$$

Integrate the left-hand side with respect to u and the right-hand side with respect to t. Substitute u back by the function y, hence we obtain

$$\frac{1}{5}\, y^5(t) + \frac{a_4}{4}\, y^4(t) + \frac{a_3}{3}\, y^3(t) + \frac{a_2}{2}\, y(t) + a_1\, y(t) = \frac{t^3}{3} + c.$$

This is an implicit form for the solution y of the problem. The solution is the root of a polynomial degree five for all possible values of the polynomial coefficients. But it has been proven that there is no formula for the roots of a general polynomial degree bigger or equal five. We conclude that that there is no explicit expression for solutions y of Eq. (1.6.5). $\lhd$

We now give an example of the statement in (ii). We consider a differential equation defined by a function f that does not satisfy one of the hypothesis in Theorem 1.6.2. The function values $f(t, u)$ have a discontinuity at a line in the (t, u) plane where the initial condition for the initial value problem is given. We then show that *such initial value problem has two solutions* instead of a unique solution.

EXAMPLE 1.6.6: Find every solution y of the initial value problem

$$y'(t) = y^{1/3}(t), \qquad y(0) = 0. \tag{1.6.6}$$

REMARK: The equation above is nonlinear, separable, and $f(t,u) = u^{1/3}$ has derivative

$$\partial_u f = \frac{1}{3}\,\frac{1}{u^{2/3}}.$$

Since the function $\partial_u f$ is not continuous at $u = 0$, it does not satisfies the Lipschitz condition in Theorem 1.6.2 on any domain of the form $S = [-a,a] \times [-a,a]$ with $a > 0$.

SOLUTION: The solution to the initial value problem in Eq. (1.6.6) exists but it is not unique, since we now show that it has two solutions. The first solution is

$$y_1(t) = 0.$$

The second solution can be computed as using the ideas from separable equations, that is,

$$\int \big[y(t)\big]^{-1/3}\, y'(t)\, dt = \int dt + c_0.$$

Then, the substitution $u = y(t)$, with $du = y'(t)\,dt$, implies that

$$\int u^{-1/3}\, du = \int dt + c_0.$$

Integrate and substitute back the function y. The result is

$$\frac{3}{2}\big[y(t)\big]^{2/3} = t + c_0 \quad \Rightarrow \quad y(t) = \Big[\frac{2}{3}(t + c_0)\Big]^{3/2}.$$

The initial condition above implies

$$0 = y(0) = \Big(\frac{2}{3}c_0\Big)^{3/2} \quad \Rightarrow \quad c_0 = 0,$$

so the second solution is:

$$y_2(t) = \Big(\frac{2}{3}t\Big)^{3/2}.$$

◁

Finally, an example of the statement in (iii). In this example we have an equation with *solutions defined in a domain that depends on the initial data.*

EXAMPLE 1.6.7: Find the solution y to the initial value problem

$$y'(t) = y^2(t), \qquad y(0) = y_0.$$

SOLUTION: This is a nonlinear separable equation, so we can again apply the ideas in Sect. 1.3. We first find all solutions of the differential equation,

$$\int \frac{y'(t)\,dt}{y^2(t)} = \int dt + c_0 \quad \Rightarrow \quad -\frac{1}{y(t)} = t + c_0 \quad \Rightarrow \quad y(t) = -\frac{1}{c_0 + t}.$$

We now use the initial condition in the last expression above,

$$y_0 = y(0) = -\frac{1}{c_0} \quad \Rightarrow \quad c_0 = -\frac{1}{y_0}.$$

So, the solution of the initial value problem above is:

$$y(t) = \frac{1}{\Big(\dfrac{1}{y_0} - t\Big)}.$$

This solution diverges at $t = 1/y_0$, so the domain of the solution y is not the whole real line $\mathbb{R}$. Instead, the domain is $\mathbb{R} - \{y_0\}$, so it depends on the values of the initial data y_0. ◁

In the next example we consider an equation of the form $y'(t) = f(t, y(t))$ for a particualr function f. We study the function values $f(t,u)$ and show the regions on the tu-plane where the hypotheses in Theorem 1.6.2 are not satisfied.

Example 1.6.8: Consider the nonlinear initial value problem

$$y'(t) = \frac{1}{(t-1)(t+1)(y(t)-2)(y(t)+3)}, \qquad y(t_0) = y_0. \tag{1.6.7}$$

Find the regions on the plane where the hypotheses in Theorem 1.6.2 are not satisfied.

Solution: In this case the function f is given by:

$$f(t,u) = \frac{1}{(t-1)(t+1)(u-2)(u+3)}, \tag{1.6.8}$$

so f is not defined on the lines

$$t = 1, \qquad t = -1, \qquad u = 2, \qquad u = -3.$$

See Fig. 6. For example, in the case that the initial data is $t_0 = 0$, $y_0 = 1$, then Theorem 1.6.2 implies that there exists a unique solution on any region $\hat{R}$ contained in the rectangle $R = (-1,1) \times (-3,2)$. If the initial data for the initial value problem in Eq. (1.6.7) is $t = 0$, $y_0 = 2$, then the hypotheses of Theorem 1.6.2 are not satisfied. ◁

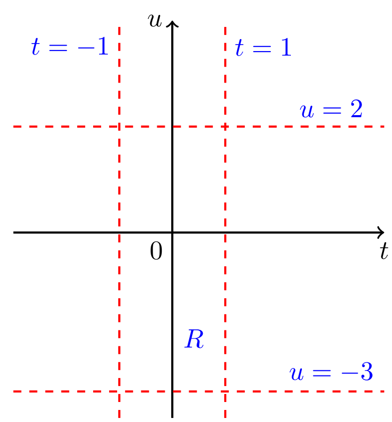


Figure 6. Red regions where f in Eq. (1.6.8) is not defined.

Summary: Both Theorems 1.2.4 and 1.6.2 state that there exist solutions to linear and nonlinear differential equations, respectively. However, Theorem 1.2.4 provides more information about the solutions to a reduced type of equations, linear problems; while Theorem 1.6.2 provides less information about solutions to wider type of equations, linear and nonlinear.

- Initial Value Problem for Linear Differential Equations
  - (a) There is an explicit expression for all the solutions.
  - (b) For every initial condition $y_0 \in \mathbb{R}$ there exists a unique solution.
  - (c) The domain of all solutionis independent of the initial condition $y_0 \in \mathbb{R}$.
- Initial Value Problem for Nonlinear Differential Equations
  - (i) There is no general explicit expression for all solutions $y(t)$.
  - (ii) Solutions may be nonunique at points $(t,u) \in \mathbb{R}^2$ where $\partial_u f$ is not continuous.
  - (iii) The domain of the solution may depend on the initial data y_0.

1.6.3. **Direction Fields.** Nonlinear differential equations are more difficult to solve that the linear ones. Then it is important to develop methods to obtain any type of information from the solution of a differential equation without having to actually solve the equation. One of such methods is based on direction fields. Consider a differential equation

$$y'(t) = f(t, y(t)).$$

One does not need to solve the differential equation above to have a qualitative idea of the solution. We only need to recall that $y'(t)$ represents the slope of the tangent line to the graph of function y at the point $(t, y(t))$ in the ty-plane. Therefore, the differential equation above provides all these slopes, $f(t, y(t))$, for every point $(t, y(t))$ in the ty-plane. So here coems the key idea to construct a direction field. *Graph the function values $f(t,y)$ on the ty-plane, not as points, but as slopes of small segments.*

Definition 1.6.3. *A* ***direction field*** *for the differential equation $y'(t) = f(t, y(t))$ is the graph on the ty-plane of the values $f(t, y)$ as slopes of a small segments.*

EXAMPLE 1.6.9: We know that the solutions of $y' = y$ are the exponentials $y(t) = y_0\, e^t$, for any constant $y_0 \in \mathbb{R}$. The graph of these solution is simple. So is the direction field shown in Fig. 7. ◁

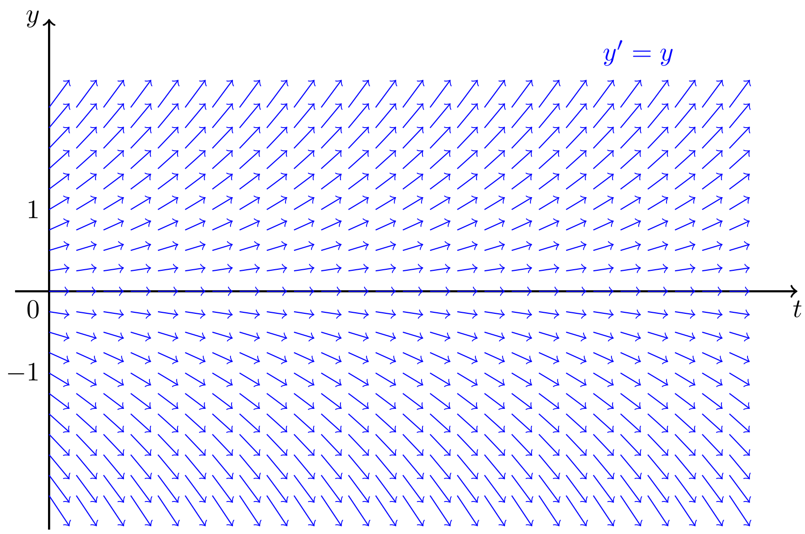


FIGURE 7. Direction field for the equation $y' = y$.

EXAMPLE 1.6.10: The equation $y' = \sin(y)$ is separable so the solutions can be computed using the ideas from § 1.3. The implicit solutions are $\ln\left|\dfrac{\csc(y_0) + \cot(y_0)}{\csc(y) + \cot(y)}\right| = t$, for any $y_0 \in \mathbb{R}$. The graphs of these solutions are not simple to do. But the direction field is simpler to plot and can be seen in Fig. 8. ◁

EXAMPLE 1.6.11: We do not need to compute the explicit solution of $y' = 2\cos(t)\cos(y)$ to have a qualitative idea of its solutions. The direction field can be seen in Fig. 9. ◁

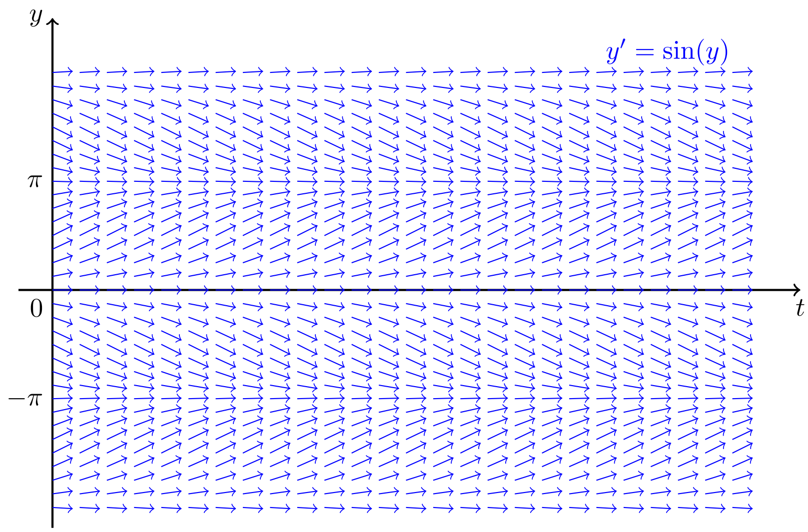


Figure 8. Direction field for the equation $y' = \sin(y)$.

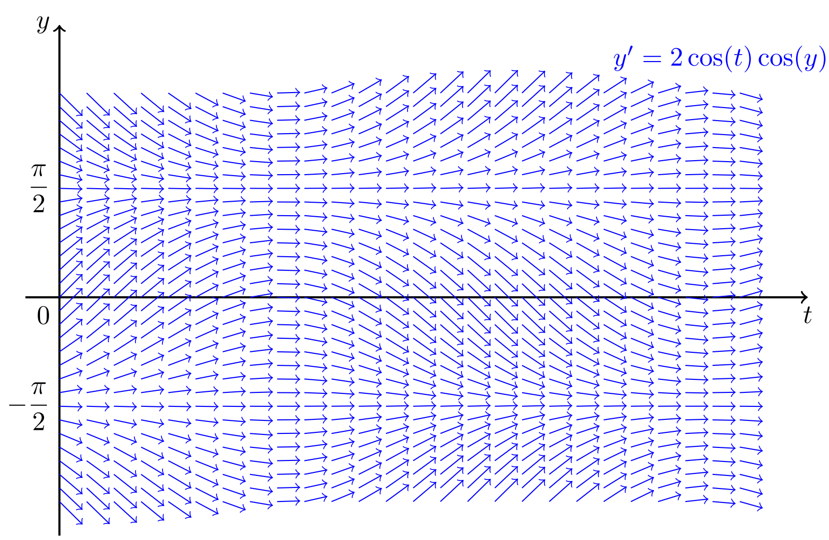


Figure 9. Direction field for the equation $y' = 2\cos(t)\cos(y)$.

1.6.4. **Exercises.**

1.6.1.- By looking at the equation coefficients, find a domain where the solution of the initial value problem below exists,

(a) $(t^2 - 4)\, y' + 2\ln(t)\, y = 3t$, and initial condition $y(1) = -2$.

(b) $y' = \dfrac{y}{t(t-3)}$, and initial condition $y(-1) = 2$.

1.6.2.- State where in the plane with points (t, y) the hypothesis of Theorem 1.6.2 are not satisfied.

(a) $y' = \dfrac{y^2}{2t - 3y}$.

(b) $y' = \sqrt{1 - t^2 - y^2}$.

1.6.3.- Find the domain where the solution of the initial value problems below is well-defined.

(a)

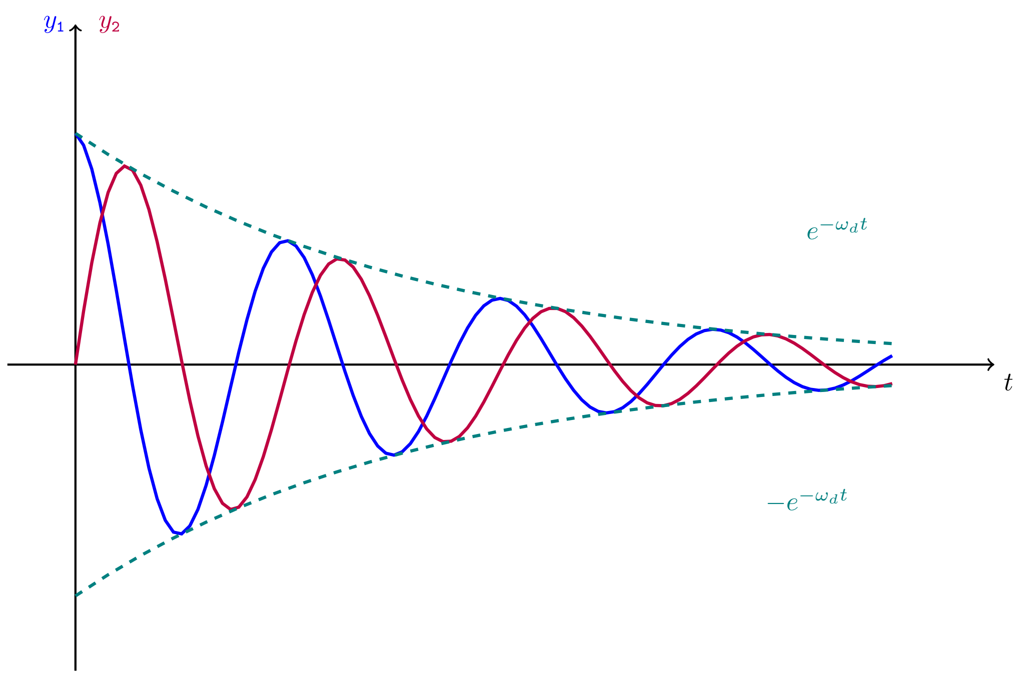
$$y' = \frac{-4t}{y}, \quad y(0) = y_0 > 0.$$

(b)

$$y' = 2ty^2, \quad y(0) = y_0 > 0.$$

Chapter 2. Second Order Linear Equations

Newton's second law of motion, $ma = f$, is maybe one of the first differential equations written. This is a second order equation, since the acceleration is the second time derivative of the particle position function. Second order differential equations are more difficult to solve than first order equations. In § 2.1 we compare results on linear first and second order equations. While there is an explicit formula for all solutions to first order linear equations, not such formula exists for all solutions to second order linear equations. The most one can get is the result in Theorem 2.1.7. In § 2.2 we introduce the Reduction Order Method to find a new solution of a second order equation if we already know one solution of the equation. In § 2.3 we find explicit formulas for all solutions to linear second order equations that are both homogeneous and with constant coefficients. These formulas are generalized to nonhomogeneous equations in § 2.4. In § 2.5 we describe a few physical systems described by second order linear differential equations.



2.1. Variable Coefficients

We studied first order linear equations in § 1.1-1.2. We obtained a formula for all solutions to these equations. We could say that we know all that can be known about the solutions to first order linear equations. This is not the case for solutions to second order linear equations. We do not have a general formula for all solutions to these equations. In this section we present two main results, which are the closer we can get to a formula for solutions to second order linear equations. Theorem 2.1.2 states that there exist solutions to second order linear equations when the equation coefficients are continuous functions, and these solutions have two free parameters that can be fixed by appropriate initial conditions. This is pretty far from having a formula for all solutions. Theorem 2.1.7 applies to homogeneous equations only. We have given up the case with nonzero source. This result says that to know all solutions to a second order linear homogeneous differential equation we need to know only two solutions that are not proportional to each other. Knowing two such solutions is equivalent to knowing them all. This is the closer we can get to a formula for all solutions. We need to find two solutions that are not proportional to each other. And this is for homogeneous equations only. The proof of the first result can be done with a Picard iteration, and it is left for a later section. The proof of the second theorem is also involved, but we do it in this section. We need to introduce a Wronskian function and prove other results, including Abel's Theorem.

2.1.1. The Initial Value Problem.

We start with a definition of second order linear differential equations. After a few examples we state the first of the main results, Theorem 2.1.2, about existence and uniqueness of solutions to an initial value problem in the case that the equation coefficients are continuous functions.

Definition 2.1.1. *A* ***second order linear differential equation*** *in the unknown y is*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \tag{2.1.1}$$

where a_1, a_0, $b : I \to \mathbb{R}$ are given functions on the interval $I \subset \mathbb{R}$. Equation (2.1.1) is called ***homogeneous*** *iff the source $b(t) = 0$ for all $t \in \mathbb{R}$. Equation (2.1.1) is called of* ***constant coefficients*** *iff a_1 and a_0 are constants; otherwise the equation is called of* ***variable coefficients****.*

Remark: The notion of an homogeneous equation presented here is different from the Euler homogeneous equations we studied in Section 1.3.

Example 2.1.1:

(a) A second order, linear, homogeneous, constant coefficients equation is

$$y'' + 5y' + 6 = 0.$$

(b) A second order, linear, nonhomogeneous, constant coefficients, equation is

$$y'' - 3y' + y = \cos(3t).$$

(c) A second order, linear, nonhomogeneous, variable coefficients equation is

$$y'' + 2t\, y' - \ln(t)\, y = e^{3t}.$$

(d) Newton's second law of motion for point particles of mass m moving in one space dimension under a force f is given by

$$m\, y''(t) = f(t).$$

This equation that says that "mass times acceleration equal force." The acceleration is the second time derivative of the position function y. ◁

Example 2.1.2: Find the differential equation satisfied by the family of functions

$$y(t) = c_1\, e^{4t} + c_2\, e^{-4t}.$$

where c_1, c_2 are arbitrary constants.

Solution: From the definition of y compute c_1,

$$c_1 = y\, e^{-4t} - c_2\, e^{-8t}.$$

Now compute the derivative of function y

$$y' = 4c_1 e^{4t} - 4c_2\, e^{-4t},$$

Replace c_1 from the first equation above into the expression for y',

$$y' = 4(y\, e^{-4t} - c_2\, e^{-8t})e^{4t} - 4c_2 e^{-4t} \quad \Rightarrow \quad y' = 4y + (-4-4)c_2 e^{-4t},$$

so we get an expression for c_2 in terms of y and y',

$$c_2 = \frac{1}{8}(4y - y')\, e^{4t}$$

We can now compute c_1 in terms of y and y',

$$c_1 = y\, e^{-4t} - \frac{1}{8}(4y - y')e^{4t} e^{-8t} \quad \Rightarrow \quad c_1 = \frac{1}{8}\,(4y + y')\, e^{-4t}.$$

We can now take the expression of either c_1 or c_2 and compute one more derivative. We choose c_2,

$$0 = c_2' = \frac{1}{2}(4y - y')e^{4t} + \frac{1}{8}(4y' - y'')\, e^{4t} \quad \Rightarrow \quad 4(4y - y') + (4y' - y'') = 0$$

which gives us the following second order linear differential equation for y,

$$y'' - 16\, y = 0.$$

◁

Example 2.1.3: Find the differential equation satisfied by the family of functions

$$y(x) = c_1\, x + c_2\, x^2.$$

where c_1, c_2 are arbitrary constants.

Solution: Compute the derivative of function y

$$y'(x) = c_1 + 2c_2\, x,$$

From here it is simple to get c_1,

$$c_1 = y' - 2c_2\, x.$$

Use this expression for c_1 in the expression for y,

$$y = (y' - 2c_2\, x)\, x + c_2\, x^2 = x\, y' - c_2\, x^2 \quad \Rightarrow \quad c_2 = \frac{y'}{x} - \frac{y}{x^2}.$$

Therefore we get for c_1 the expression

$$c_1 = y' - 2\Big(\frac{y'}{x} - \frac{y}{x^2}\Big)\, x = y' - 2\, y' + \frac{2y}{x} \quad \Rightarrow \quad c_1 = -y' + \frac{2y}{x}.$$

To obtain an equation for y we compute its second derivative, and replace in that derivative the formulas for the constants c_1 and c_2. In this particular example we only need c_2,

$$y'' = 2c_2 = 2\Big(\frac{y'}{x} - \frac{y}{x^2}\Big) \quad \Rightarrow \quad y'' - \frac{2}{x}\, y' + \frac{2}{x^2}\, y = 0.$$

◁

Here is the first of the two main results in this section. Second order linear differential equations have solutions in the case that the equation coefficients are continuous functions. Since the solution is unique when we specify two extra conditions, called initial conditions, we infer that a general solution must have two arbitrary integration constants.

Theorem 2.1.2 (Variable Coefficients). *If the functions a_1, a_0, $b : I \to \mathbb{R}$ are continuous on a closed interval $I \subset \mathbb{R}$, $t_0 \in I$, and y_0, $y_1 \in \mathbb{R}$ are any constants, then there exists a unique solution $y : I \to \mathbb{R}$ to the initial value problem*

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{2.1.2}$$

Remark: The fixed point argument used in the proof of Picard-Lindelöf's Theorem 1.6.2 can be extended to prove Theorem 2.1.2. This proof will be presented later on.

EXAMPLE 2.1.4: Find the longest interval $I \in \mathbb{R}$ such that there exists a unique solution to the initial value problem

$$(t-1)y'' - 3ty' + 4y = t(t-1), \qquad y(-2) = 2, \qquad y'(-2) = 1.$$

SOLUTION: We first write the equation above in the form given in Theorem 2.1.2,

$$y'' - \frac{3t}{t-1}\, y' + \frac{4}{t-1}\, y = t.$$

The intervals where the hypotheses in the Theorem above are satisfied, that is, where the equation coefficients are continuous, are $I_1 = (-\infty, 1)$ and $I_2 = (1, \infty)$. Since the initial condition belongs to I_1, the solution domain is

$$I_1 = (-\infty, 1).$$

$\lhd$

2.1.2. **Homogeneous Equations.** We need to simplify the problem to get further in its solution. From now on in this section we study homogeneous equations only. Once we learn properties of solutions to homogeneous equations we can get back at the nonhomogeneous case. But before getting into homogeneous equations, we introduce a new notation to write differential equations. This is a shorter, more economical, notation. Given two functions a_1, a_0, introduce the function L acting on a function y, as follows,

$$L(y) = y'' + a_1(t)\, y' + a_0(t)\, y. \tag{2.1.3}$$

The function L acts on the function y and the result is another function, given by Eq. (2.1.3).

EXAMPLE 2.1.5: Compute the operator $L(y) = t\, y'' + 2y' - \dfrac{8}{t}\, y$ acting on $y(t) = t^3$.

SOLUTION: Since $y(t) = t^3$, then $y'(t) = 3t^2$ and $y''(t) = 6t$, hence

$$L(t^3) = t\,(6t) + 2(3t^2) - \frac{8}{t}\, t^3 \quad \Rightarrow \quad L(t^3) = 4t^2.$$

The function L acts on the function $y(t) = t^3$ and the result is the function $L(t^3) = 4t^2$. $\lhd$

To emphasize that L is a function that acts on other functions, instead of acting on numbers, like usual functions, is that L is also called a *functional*, or an *operator*. As shown in the Example above, operators may involve computing derivatives of the function they act upon. So, operators are useful to write differential equations in a compact notation, since

$$y'' + a_1(t)\, y' + a_0(t)\, y = f(t)$$

can be written using the operator $L(y) = y'' + a_1(t)\,y' + a_0(t)\,y$ as

$$L(y) = f.$$

An important type of operators is called linear operators.

Definition 2.1.3. *An operator L is called a* ***linear operator*** *iff for every pair of functions y_1, y_2 and constants c_1, c_2 holds true that*

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2). \tag{2.1.4}$$

In this Section we work with linear operators, as the following result shows.

Theorem 2.1.4 (Linear Operator). *The operator $L(y) = y'' + a_1\,y' + a_0\,y$, where a_1, a_0 are continuous functions and y is a twice differentiable function, is a linear operator.*

Proof of Theorem 2.1.4: This is a straightforward calculation:

$$L(c_1y_1 + c_2y_2) = (c_1y_1 + c_2y_2)'' + a_1\,(c_1y_1 + c_2y_2)' + a_0\,(c_1y_1 + c_2y_2).$$

Recall that derivations is a linear operation and then reorder terms in the following way,

$$L(c_1y_1 + c_2y_2) = \big(c_1y_1'' + a_1\,c_1y_1' + a_0\,c_1y_1\big) + \big(c_2y_2'' + a_1\,c_2y_2' + a_0\,c_2y_2\big).$$

Introduce the definition of L back on the right-hand side. We then conclude that

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$

This establishes the Theorem. □

The linearity of an operator L translates into the superposition property of the solutions to the homogeneous equation $L(y) = 0$.

Theorem 2.1.5 (Superposition). *If L is a linear operator and y_1, y_2 are solutions of the homogeneous equations $L(y_1) = 0$, $L(y_2) = 0$, then for every constants c_1, c_2 holds true that $L(c_1\,y_1 + c_2\,y_2) = 0$.*

Remark: This result is not true for nonhomogeneous equations.

Proof of Theorem 2.1.5: Verify that the function $y = c_1y_1 + c_2y_2$ satisfies $L(y) = 0$ for every constants c_1, c_2, that is,

$$L(y) = L(c_1y_1 + c_2y_2) = c_1\,L(y_1) + c_2\,L(y_2) = c_1\,0 + c_2\,0 = 0.$$

This establishes the Theorem. □

We now introduce the notion of linearly dependent and linearly independent functions.

Definition 2.1.6. *Two continuous functions y_1, $y_2 : I \to \mathbb{R}$ are called* ***linearly dependent*** *on the interval I iff there exists a constant c such that for all $t \in I$ holds*

$$y_1(t) = c\,y_2(t).$$

Two functions are called ***linearly independent*** *on I iff they are not linearly dependent.*

In words only, two functions are linearly dependent on an interval iff the functions are proportional to each other on that interval, otherwise they are linearly independent.

Remark: The function $y_1 = 0$ is proportional to every other function y_2, since holds $y_1 = 0 = 0\,y_2$. Hence, any set containing the zero function is linearly dependent.

The definitions of linearly dependent or independent functions found in the literature are equivalent to the definition given here, but they are worded in a slight different way. One usually finds in the literature that two functions are called linearly dependent on the interval I iff there exist constants c_1, c_2, not both zero, such that for all $t \in I$ holds

$$c_1y_1(t) + c_2y_2(t) = 0.$$

Two functions are called linearly independent on the interval I iff they are not linearly dependent, that is, the only constants c_1 and c_2 that for all $t \in I$ satisfy the equation

$$c_1 y_1(t) + c_2 y_2(t) = 0$$

are the constants $c_1 = c_2 = 0$. This latter wording makes it simple to generalize these definitions to an arbitrary number of functions.

EXAMPLE 2.1.6:

(a) Show that $y_1(t) = \sin(t)$, $y_2(t) = 2\sin(t)$ are linearly dependent.
(b) Show that $y_1(t) = \sin(t)$, $y_2(t) = t\sin(t)$ are linearly independent.

SOLUTION:

Part (a): This is trivial, since $2y_1(t) - y_2(t) = 0$.

Part (b): Find constants c_1, c_2 such that for all $t \in \mathbb{R}$ holds

$$c_1 \sin(t) + c_2 t \sin(t) = 0.$$

Evaluating at $t = \pi/2$ and $t = 3\pi/2$ we obtain

$$c_1 + \frac{\pi}{2} c_2 = 0, \quad c_1 + \frac{3\pi}{2} c_2 = 0 \quad \Rightarrow \quad c_1 = 0, \quad c_2 = 0.$$

We conclude: The functions y_1 and y_2 are linearly independent. ◁

We now introduce the second main result in this section. If you know two linearly independent solutions to a second order linear homogeneous differential equation, then you actually know all possible solutions to that equation. Any other solution is just a linear combination of the previous two solutions. We repeat that the equation must be homogeneous. This is the closer we can get to a general formula for solutions to second order linear homogeneous differential equations.

Theorem 2.1.7 (General Solution). *If y_1 and y_2 are linearly independent solutions of the equation $L(y) = 0$ on an interval $I \subset \mathbb{R}$, where $L(y) = y'' + a_1\, y' + a_0\, y$, and a_1, a_2 are continuous functions on I, then there exist unique constants c_1, c_2 such that every solution y of the differential equation $L(y) = 0$ on I can be written as a linear combination*

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t).$$

Before we prove Theorem 2.1.7, it is convenient to state the following the definitions, which come out naturally from this Theorem.

Definition 2.1.8.

*(a) The functions y_1 and y_2 are **fundamental solutions** of the equation $L(y) = 0$ iff holds that $L(y_1) = 0$, $L(y_2) = 0$ and y_1, y_2 are linearly independent.*

*(b) The **general solution** of the homogeneous equation $L(y) = 0$ is a two-parameter family of functions y_{gen} given by*

$$y_{\text{gen}}(t) = c_1\, y_1(t) + c_2\, y_2(t),$$

where the arbitrary constants c_1, c_2 are the parameters of the family, and y_1, y_2 are fundamental solutions of $L(y) = 0$.

EXAMPLE 2.1.7: Show that $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions to the equation

$$y'' + y' - 2y = 0.$$

SOLUTION: We first show that y_1 and y_2 are solutions to the differential equation, since

$$L(y_1) = y_1'' + y_1' - 2y_1 = e^t + e^t - 2e^t = (1 + 1 - 2)e^t = 0,$$

$$L(y_2) = y_2'' + y_2' - 2y_2 = 4\,e^{-2t} - 2\,e^{-2t} - 2e^{-2t} = (4-2-2)e^{-2t} = 0.$$

It is not difficult to see that y_1 and y_2 are linearly independent. It is clear that they are not proportional to each other. A proof of that statement is the following: Find the constants c_1 and c_2 such that

$$0 = c_1\,y_1 + c_2\,y_2 = c_1\,e^t + c_2\,e^{-2t} \quad t \in \mathbb{R} \quad \Rightarrow \quad 0 = c_1\,e^t - 2c_2\,e^{-2t}$$

The second equation is the derivative of the first one. Take $t = 0$ in both equations,

$$0 = c_1 + c_2, \quad 0 = c_1 - 2c_2 \quad \Rightarrow \quad c_1 = c_2 = 0.$$

We conclude that y_1 and y_2 are fundamental solutions to the differential equation above.$\lhd$

Remark: The fundamental solutions to the equation above are not unique. For example, show that another set of fundamental solutions to the equation above is given by,

$$y_1(t) = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}, \qquad y_2(t) = \frac{1}{3}\big(e^t - e^{-2t}\big).$$

To prove Theorem 2.1.7 we need to introduce the Wronskian function and to verify some of its properties. The Wronskian function is studied in the following Subsection and Abel's Theorem is proved. Once that is done we can say that the proof of Theorem 2.1.7 is complete.

Proof of Theorem 2.1.7: We need to show that, given any fundamental solution pair, y_1, y_2, any other solution y to the homogeneous equation $L(y) = 0$ must be a unique linear combination of the fundamental solutions,

$$y(t) = c_1\,y_1(t) + c_2\,y_2(t), \tag{2.1.5}$$

for appropriately chosen constants c_1, c_2.

First, the superposition property implies that the function y above is solution of the homogeneous equation $L(y) = 0$ for every pair of constants c_1, c_2.

Second, given a function y, if there exist constants c_1, c_2 such that Eq. (2.1.5) holds, then these constants are unique. The reason is that functions y_1, y_2 are linearly independent. This can be seen from the following argument. If there are another constants $\tilde{c}_1$, $\tilde{c}_2$ so that

$$y(t) = \tilde{c}_1\,y_1(t) + \tilde{c}_2\,y_2(t),$$

then subtract the expression above from Eq. (2.1.5),

$$0 = (c_1 - \tilde{c}_1)\,y_1 + (c_2 - \tilde{c}_2)\,y_2 \quad \Rightarrow \quad c_1 - \tilde{c}_1 = 0, \quad c_2 - \tilde{c}_2 = 0,$$

where we used that y_1, y_2 are linearly independent. This second part of the proof can be obtained from the part three below, but I think it is better to highlight it here.

So we only need to show that the expression in Eq. (2.1.5) contains all solutions. We need to show that we are not missing any other solution. In this third part of the argument enters Theorem 2.1.2. This Theorem says that, in the case of homogeneous equations, the initial value problem

$$L(y) = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = \hat{y}_1,$$

always has a unique solution. That means, a good parametrization of all solutions to the differential equation $L(y) = 0$ is given by the two constants, y_0, $\hat{y}_1$ in the initial condition. To finish the proof of Theorem 2.1.7 we need to show that the constants c_1 and c_2 are also good to parametrize all solutions to the equation $L(y) = 0$. One way to show this, is to find an invertible map from the constants y_0, $\hat{y}_1$, which we know parametrize all solutions, to the constants c_1, c_2. The map itself is simple to find,

$$\begin{aligned} y_0 &= c_1\, y_1(t_0) + c_2\, y_2(t_0) \\ \hat{y}_1 &= c_1\, y_1'(t_0) + c_2\, y_2'(t_0). \end{aligned}$$

We now need to show that this map is invertible. From linear algebra we know that this map acting on c_1, c_2 is invertible iff the determinant of the coefficient matrix is nonzero,

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0.$$

This leads us to investigate the function

$$W_{y_1,y_2}(t) = y_1(t)\, y_2'(t) - y_1'(t) y_2(t)$$

This function is called the Wronskian of the two functions y_1, y_2. So, now the proof of the Theorem rests in the answer to the question: Given any two linearly independent solutions y_1, y_2 of the homogeneous equation $L(y) = 0$ in the interval I, is it true that their Wronskian at every point $t \in I$ is nonzero? We prove in the next Subsection 2.1.3, Corollary 2.1.13, that the answer is "yes". This establishes the Theorem. □

2.1.3. **The Wronskian Function.** We now introduce a function that provides important information about the linear dependency of two functions y_1, y_2. This function, W, is called the Wronskian to honor the polish scientist Josef Wronski, who first introduced this function in 1821 while studying a different problem.

Definition 2.1.9. *The **Wronskian** of the differentiable functions y_1, y_2 is the function*

$$W_{y_1 y_2}(t) = y_1(t) y_2'(t) - y_1'(t) y_2(t).$$

Remark: Introducing the matrix-valued function $A(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$ the Wronskian can be written using the determinant of that 2×2 matrix, $W_{y_1 y_2}(t) = \det\big(A(t)\big)$. An alternative notation is: $W_{y_1 y_2} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

EXAMPLE 2.1.8: Find the Wronskian of the functions:
(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$. (ld)
(b) $y_1(t) = \sin(t)$ and $y_2(t) = t\sin(t)$. (li)

SOLUTION:
Part (a): By the definition of the Wronskian:

$$W_{y_1 y_2}(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix} = \sin(t) 2\cos(t) - \cos(t) 2\sin(t)$$

We conclude that $W_{y_1 y_2}(t) = 0$. Notice that y_1 and y_2 are linearly dependent.

Part (b): Again, by the definition of the Wronskian:

$$W_{y_1 y_2}(t) = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix} = \sin(t)\big[\sin(t) + t\cos(t)\big] - \cos(t) t\sin(t).$$

We conclude that $W_{y_1 y_2}(t) = \sin^2(t)$. Notice that y_1 and y_2 are linearly independent. ◁

It is simple to prove the following relation between the Wronskian of two functions and the linear dependency of these two functions.

Theorem 2.1.10 (Wronskian I). *If the functions y_1, $y_2 : I \to \mathbb{R}$ are linearly dependent, then their Wronskian function vanishes identically on the interval I.*

Proof of Theorem 2.1.10: Since the functions y_1, y_2 are linearly dependent, there exists a nonzero constant c such that $y_1 = c\, y_2$; hence holds,

$$W_{y_1 y_2} = y_1\, y_2' - y_1'\, y_2 = (c\, y_2)\, y_2' - (c\, y_2)'\, y_2 = 0.$$

This establishes the Theorem. □

Remark: The converse statement to Theorem 2.1.10 is false. If $W_{y_1 y_2}(t) = 0$ for all $t \in I$, that ***does not*** imply that y_1 and y_2 are linearly dependent. Just consider the example

$$y_1(t) = t^2, \quad \text{and} \quad y_2(t) = |t|\, t,$$

for $t \in \mathbb{R}$. Both functions are differentiable in $\mathbb{R}$, so their Wronskian can be computed. It is not hard to see that these functions have $W_{y_1 y_2}(t) = 0$ for $t \in \mathbb{R}$. However, they are not linearly dependent, since $y_1(t) = -y_2(t)$ for $t < 0$, but $y_1(t) = y_2(t)$ for $t > 0$.

Often in the literature one finds the negative of Theorem 2.1.10. This is not new for us now, since it is equivalent to Theorem 2.1.10. We show it as a Corollary of that Theorem.

Corollary 2.1.11 (Wronskian I). *If the Wronskian $W_{y_1 y_2}(t_0) \neq 0$ at a single point $t_0 \in I$, then the functions y_1, $y_2 : I \to \mathbb{R}$ are linearly independent.*

By looking at the Corollary is clear that we need some sort of converse statement to this Corollary to finish the proof of Theorem 2.1.7. However, as we stated in the Remark, the converse statement for Theorem 2.1.10 is not true, hence the same holds for the Corollary above. One needs to know something else about the functions y_1, y_2, besides their zero Wronskian, to conclude that these functions are linearly dependent. In our case, this extra hypothesis is that functions y_1, y_2 are solutions to the same homogeneous differential equation. One then obtains a result much stronger than the converse of Theorem 2.1.10.

Theorem 2.1.12 (Wronskian II). *Let y_1, $y_2 : I \to \mathbb{R}$ be both solutions of $L(y) = 0$ on I. If there exists one point $t_0 \in I$ such that $W_{y_1 y_2}(t_0) = 0$, then y_1 y_2 are linearly dependent.*

Remark: Since in the Theorem above we conclude that the functions y_1, y_2 are linearly dependent, then Theorem 2.1.10 says that their Wronskian vanishes identically in I.

We present the negative of this statement as the following Corollary, since it is precisely this Corollary what we need to finish the proof of Theorem 2.1.7.

Corollary 2.1.13 (Wronskian II). *Let y_1, $y_2 : I \to \mathbb{R}$ be both solutions of $L(y) = 0$ on I. If y_1 y_2 are linearly independent on I, then their Wronskian $W_{y_1 y_2}(t) \neq 0$ for all $t \in I$.*

Since to prove Theorem 2.1.12 is to prove the Corollary 2.1.13, we focus on the Theorem.
Proof of Theorem 2.1.12: The first step is to use Abel's Theorem, which is stated an proven below. Abel's Theorem says that, if the Wronskian $W_{y_1 y_2}(t_0) = 0$, then $W_{y_1 y_2}(t) = 0$ for all $t \in I$. Knowing that the Wronskian vanishes identically on I, we can write down,

$$y_1\, y_2' - y_1'\, y_2 = 0,$$

on I. If either y_1 or y_2 is the function zero, then the set is linearly dependent. So we can assume that both are not identically zero. Let's assume there exists $t_1 \in I$ such that $y_1(t_1) \neq 0$. By continuity, y_1 is nonzero in an open neighborhood $I_1 \subset I$ of t_1. So in that neighborhood we can divide the equation above by y_1^2,

$$\frac{y_1\, y_2' - y_1'\, y_2}{y_1^2} = 0 \quad \Rightarrow \quad \Big(\frac{y_2}{y_1}\Big)' = 0 \quad \Rightarrow \quad \frac{y_2}{y_1} = c, \qquad \text{on } I_1,$$

where $c \in \mathbb{R}$ is an arbitrary constant. So we conclude that y_1 is proportional to y_2 on the open set I_1. That means that the function $y(t) = y_2(t) - c\, y_1(t)$, satisfies

$$L(y) = 0, \qquad y(t_1) = 0, \quad y'(t_1) = 0.$$

Therefore, the existence and uniqueness Theorem 2.1.2 says that $y(t) = 0$ for all $t \in I$. This finally shows that y_1 and y_2 are linearly dependent. This establishes the Theorem. □

2.1.4. **Abel's Theorem.** So we only need to put the final piece of this puzzle. We now state and prove Abel's Theorem on the Wronskian of two solutions to an homogeneous differential equation.

Theorem 2.1.14 (Abel). *If y_1, y_2 are twice continuously differentiable solutions of*

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{2.1.6}$$

where a_1, a_0 are continuous on $I \subset \mathbb{R}$, then the Wronskian $W_{y_1 y_2}$ satisfies

$$W'_{y_1 y_2} + a_1(t)\, W_{y_1 y_2} = 0.$$

Therefore, for any $t_0 \in I$, the Wronskian $W_{y_1 y_2}$ is given by the expression

$$W_{y_1 y_2}(t) = W_0\, e^{-A_1(t)},$$

where $W_0 = W_{y_1 y_2}(t_0)$ and $A_1(t) = \displaystyle\int_{t_0}^{t} a_1(s)\, ds$.

Before the proof of Abel's Theorem, we show an application.

Example 2.1.9: Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

Solution: Notice that we do not known the explicit expression for the solutions. Nevertheless, Theorem 2.1.14 says that we can compute their Wronskian. First, we have to rewrite the differential equation in the form given in that Theorem, namely,

$$y'' - \Big(\frac{2}{t} + 1\Big)\, y' + \Big(\frac{2}{t^2} + \frac{1}{t}\Big)\, y = 0.$$

Then, Theorem 2.1.14 says that the Wronskian satisfies the differential equation

$$W'_{y_1 y_2}(t) - \Big(\frac{2}{t} + 1\Big)\, W_{y_1 y_2}(t) = 0.$$

This is a first order, linear equation for $W_{y_1 y_2}$, so its solution can be computed using the method of integrating factors. That is, first compute the integral

$$-\int_{t_0}^{t} \Big(\frac{2}{s} + 1\Big)\, ds = -2 \ln\Big(\frac{t}{t_0}\Big) - (t - t_0)$$
$$= \ln\Big(\frac{t_0^2}{t^2}\Big) - (t - t_0).$$

Then, the integrating factor μ is given by

$$\mu(t) = \frac{t_0^2}{t^2}\, e^{-(t-t_0)},$$

which satisfies the condition $\mu(t_0) = 1$. So the solution, $W_{y_1 y_2}$ is given by

$$\Big[\mu(t) W_{y_1 y_2}(t)\Big]' = 0 \quad \Rightarrow \quad \mu(t) W_{y_1 y_2}(t) - \mu(t_0) W_{y_1 y_2}(t_0) = 0$$

so, the solution is

$$W_{y_1 y_2}(t) = W_{y_1 y_2}(t_0)\, \frac{t^2}{t_0^2}\, e^{(t-t_0)}.$$

If we call the constant $c = W_{y_1 y_2}(t_0)/[t_0^2 e^{t_0}]$, then the Wronskian has the simpler form

$$W_{y_1 y_2}(t) = c\, t^2 e^t.$$

$\triangleleft$

Proof of Theorem 2.1.14: We start computing the derivative of the Wronskian function,

$$W'_{y_1 y_2} = \big(y_1\, y_2' - y_1'\, y_2\big)' = y_1\, y_2'' - y_1''\, y_2.$$

Recall that both y_1 and y_2 are solutions to Eq. (2.1.6), meaning,

$$y_1'' = -a_1\, y_1' - a_0\, y_1, \qquad y_2'' = -a_1\, y_2' - a_0\, y_2.$$

Replace these expressions in the formula for $W'_{y_1 y_2}$ above,

$$W'_{y_1 y_2} = y_1\,\big(-a_1\, y_2' - a_0\, y_2\big) - \big(-a_1\, y_1' - a_0\, y_1\big)\, y_2 \quad \Rightarrow \quad W'_{y_1 y_2} = -a_1\big(y_1\, y_2' - y_1'\, y_2\big)$$

So we obtain the equation

$$W'_{y_1 y_2} + a_1(t)\; W_{y_1 y_2} = 0.$$

This equation for $W_{y_1 y_2}$ is a first order linear equation; its solution can be found using the method of integrating factors, given in Section 1.1, which results is the expression in the Theorem 2.1.14. This establishes the Theorem. $\square$

2.1.5. **Exercises.**

2.1.1.- Compute the Wronskian of the following functions:

(a) $f(t) = \sin(t)$, $g(t) = \cos(t)$.
(b) $f(x) = x$, $g(x) = x\,e^x$.
(c) $f(\theta) = \cos^2(\theta)$, $g(\theta) = 1 + \cos(2\theta)$.

2.1.2.- Find the longest interval where the solution y of the initial value problems below is defined. (Do not try to solve the differential equations.)

(a) $t^2 y'' + 6y = 2t$, $y(1) = 2$, $y'(1) = 3$.
(b) $(t-6)y' + 3ty' - y = 1$, $y(3) = -1$, $y'(3) = 2$.

2.1.3.- (a) Verify that $y_1(t) = t^2$ and $y_2(t) = 1/t$ are solutions to the differential equation

$$t^2 y'' - 2y = 0, \quad t > 0.$$

(b) Show that $y(t) = a\,t^2 + \dfrac{b}{t}$ is solution of the same equation for all constants $a, b \in \mathbb{R}$.

2.1.4.- If the graph of y, solution to a second order linear differential equation $L(y(t)) = 0$ on the interval $[a, b]$, is tangent to the t-axis at any point $t_0 \in [a, b]$, then find the solution y explicitly.

2.1.5.- Can the function $y(t) = \sin(t^2)$ be solution on an open interval containing $t = 0$ of a differential equation

$$y'' + a(t)\,y' + b(t)y = 0,$$

with continuous coefficients a and b? Explain your answer.

2.1.6.- Verify whether the functions y_1, y_2 below are a fundamental set for the differential equations given below:

(a) $y_1(t) = \cos(2t)$, $y_2(t) = \sin(2t)$,
$$y'' + 4y = 0.$$
(b) $y_1(t) = e^t$, $y_2(t) = t\,e^t$,
$$y'' - 2y' + y = 0.$$
(c) $y_1(x) = x$, $y_2(t) = x\,e^x$,
$$x^2\,y'' - 2x(x+2)\,y' + (x+2)\,y = 0.$$

2.1.7.- If the Wronskian of any two solutions of the differential equation

$$y'' + p(t)\,y' + q(t)\,y = 0$$

is constant, what does this imply about the coefficients p and q?

2.1.8.- Let $y(t) = c_1\,t + c_2\,t^2$ be the general solution of a second order linear differential equation $L(y) = 0$. By eliminating the constants c_1 and c_2, find the differential equation satisfied by y.

2.2. Reduction of Order Methods

Sometimes a solution to a second order differential equation can be obtained solving two first order equations, one after the other. When that happens we say we have reduced the order of the equation. Although the equation is still second order, the two equations we solve are each one first order. We then use methods from Chapter 1 to solve the first order equations. In this section we focus on three types of differential equations where such reduction of order happens. The first two cases are usually called special second order equations and only the third case is called a reduction of order method. We follow this convention here, althought all three methods reduce the order of the original equation.

2.2.1. Special Second Order Equations.

A second order differential equation is called special when either the unknown function or the independent variable does not appear explicitly in the equation. In either case, such second order equation can be transformed in a first order equation for a new unknown function. The transformation to get the new unknown function is different on each case. One then solves the first order equation and transforms back solving another first order equation to get the original unknown function. We now start with a few definitions.

Definition 2.2.1. *A **second order** equation in the unknown function y is an equation*

$$y'' = f(t, y, y').$$

*where the function $f : \mathbb{R}^3 \to \mathbb{R}$ is given. The equation is **linear** iff function f is linear in both arguments y and y'. The second order differential equation above is **special** iff one of the following conditions hold:*

(a) $y'' = f(t, y')$, so the function y does not appear explicitly in the equation;
(b) $y'' = f(y, y')$, so the independent variable t does not appear explicitly in the equation.

It is simpler to solve special second order equations when the function y missing, case (a), than when the variable t is missing, case (b). This can be seen comparing Theorems 2.2.2 and 2.2.3.

Theorem 2.2.2 (Function y Missing). *If a second order differential equation has the form $y'' = f(t, y')$, then $v = y'$ satisfies the first order equation $v' = f(t, v)$.*

The proof is trivial, so we go directly to an example.

Example 2.2.1: Find the y solution of the second order nonlinear equation $y'' = -2t\,(y')^2$ with initial conditions $y(0) = 2$, $y'(0) = -1$.

Solution: Introduce $v = y'$. Then $v' = y''$, and

$$v' = -2t\,v^2 \quad \Rightarrow \quad \frac{v'}{v^2} = -2t \quad \Rightarrow \quad -\frac{1}{v} = -t^2 + c.$$

So, $\dfrac{1}{y'} = t^2 - c$, that is, $y' = \dfrac{1}{t^2 - c}$. The initial condition implies

$$-1 = y'(0) = -\frac{1}{c} \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad y' = \frac{1}{t^2 - 1}.$$

Then, $y = \displaystyle\int \frac{dt}{t^2 - 1} + c$. We integrate using the method of partial fractions,

$$\frac{1}{t^2 - 1} = \frac{1}{(t-1)(t+1)} = \frac{a}{(t-1)} + \frac{b}{(t+1)}.$$

Hence, $1 = a(t+1) + b(t-1)$. Evaluating at $t = 1$ and $t = -1$ we get $a = \frac{1}{2}$, $b = -\frac{1}{2}$. So

$$\frac{1}{t^2-1} = \frac{1}{2}\Big[\frac{1}{(t-1)} - \frac{1}{(t+1)}\Big].$$

Therefore, the integral is simple to do,

$$y = \frac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + c. \qquad 2 = y(0) = \frac{1}{2}(0-0) + c.$$

We conclude $y = \frac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + 2$. ◁

Special second order equations where the variable t is missing, case (b), are more complicated to solve.

Theorem 2.2.3 (Variable t Missing). *If a second order differential equation has the form*

$$y'' = f(y, y'),$$

and a solution y is invertible, with values $y(t)$ and inverse function values $t(y)$, then the function $w(y) = v(t(y))$, where $v(t) = y'(t)$, satisfies the first order equation

$$\dot{w} = \frac{f(y,w)}{w},$$

where we denoted $\dot{w} = dw/dy$.

Remark: The chain rule for the derivative of a composition of functions allows us to transform original differential equation in the independent variable t to a differential equation in the independent variable y.

Proof of Theorem 2.2.3: Introduce the notation

$$\dot{w}(y) = \frac{dw}{dy}, \quad v'(t) = \frac{dv}{dt}.$$

The differential equation in terms of v has the form $v'(t) = f(y(t), v(t))$. It is not clear how to solve it, since the function y still appears in that equation. For that reason we now introduce the function $w(y) = v(t(y))$, and we use the chain rule to find out the equation satisfied by that function w. The chain rule says,

$$\dot{w}(y) = \frac{dw}{dy}\Big|_y = \frac{dv}{dt}\Big|_{t(y)} \frac{dt}{dy}\Big|_{t(y)} = \frac{v'(t)}{y'(t)}\Big|_{t(y)} = \frac{v'(t)}{v(t)}\Big|_{t(y)} = \frac{f\big(y(t), v(t)\big)}{v(t)}\Big|_{t(y)} = \frac{f(y, w(y))}{w(y)}.$$

Therefore, we have obtained the equation for w, namely

$$\dot{w} = \frac{f(y,w)}{w}$$

This establishes the Theorem. □

EXAMPLE 2.2.2: Find a solution y to the second order equation $y'' = 2y\,y'$.

SOLUTION: The variable t does not appear in the equation. So we start introduciong the function $v(t) = y'(t)$. The equation is now given by $v'(t) = 2y(t)\,v(t)$. We look for invertible solutions y, then introduce the function $w(y) = v(t(y))$. This function satisfies

$$\dot{w}(y) = \frac{dw}{dy} = \Big(\frac{dv}{dt}\,\frac{dt}{dy}\Big)\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)}.$$

Using the differential equation,

$$\dot{w}(y) = \frac{2yv}{v}\Big|_{t(y)} \quad \Rightarrow \quad \frac{dw}{dy} = 2y \quad \Rightarrow \quad w(y) = y^2 + c.$$

Since $v(t) = w(y(t))$, we get $v(t) = y^2(t) + c$. This is a separable equation,

$$\frac{y'(t)}{y^2(t) + c} = 1.$$

Since we only need to find a solution of the equation, and the integral depends on whether $c > 0$, $c = 0$, $c < 0$, we choose (for no special reason) only one case, $c = 1$.

$$\int \frac{dy}{1 + y^2} = \int dt + c_0 \quad \Rightarrow \quad \arctan(y) = t + c_0 y(t) = \tan(t + c_0).$$

Again, for no reason, we choose $c_0 = 0$, and we conclude that one possible solution to our problem is $y(t) = \tan(t)$. ◁

Example 2.2.3: Find the solution y to the initial value problem

$$y\,y'' + 3(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = 6.$$

Solution: We start rewriting the equation in the standard form

$$y'' = -3\frac{(y')^2}{y}.$$

The variable t does not appear explicitly in the equation, so we introduce the function $v(t) = y'(t)$. The differential equation now has the form $v'(t) = -3v^2(t)/y(t)$. We look for invertible solutions y, and then we introduce the function $w(y) = v(t(y))$. Because of the chain rule for derivatives, this function satisfies

$$\dot{w}(y) = \frac{dw}{dy}(y) = \Big(\frac{dv}{dt}\,\frac{dt}{dy}\Big)\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)} \quad \Rightarrow \quad \dot{w}(y) = \frac{v'(t(y))}{w(y)}.$$

Using the differential equation on the factor v', we get

$$\dot{w}(y) = \frac{-3v^2(t(y))}{y}\,\frac{1}{w} = \frac{-3w^2}{yw} \quad \Rightarrow \quad \dot{w} = \frac{-3w}{y}.$$

This is a separable equation for function w. The problem for w also has initial conditions, which can be obtained from the initial conditions from y. Recalling the definition of inverse function,

$$y(t = 0) = 1 \quad \Leftrightarrow \quad t(y = 1) = 0.$$

Therefore,

$$w(y = 1) = v(t(y = 1)) = v(0) = y'(0) = 6,$$

where in the last step above we use the initial condition $y'(0) = 6$. Summarizing, the initial value problem for w is

$$\dot{w} = \frac{-3w}{y}, \quad w(1) = 6.$$

The equation for w is separable, so the method from § 1.3 implies that

$$\ln(w) = -3\ln(y) + c_0 = \ln(y^{-3}) + c_0 \quad \Rightarrow \quad w(y) = c_1\,y^{-3}, \quad c_1 = e^{c_0}.$$

The initial condition fixes the constant c_1, since

$$6 = w(1) = c_1 \quad \Rightarrow \quad w(y) = 6\,y^{-3}.$$

We now transform from w back to v as follows,

$$v(t) = w(y(t)) = 6\,y^{-3}(t) \quad \Rightarrow \quad y'(t) = 6y^{-3}(t).$$

This is now a first order separable equation for y. Again the method from § 1.3 imply that

$$y^3\,y' = 6 \quad \Rightarrow \quad \frac{y^4}{4} = 6t + c_2$$

The initial condition for y fixes the constant c_2, since

$$1 = y(0) \quad \Rightarrow \quad \frac{1}{4} = 0 + c_2 \quad \Rightarrow \quad \frac{y^4}{4} = 6t + \frac{1}{4}.$$

So we conclude that the solution y to the initial value problem is

$$y(t) = (24t + 1)^4.$$

$\lhd$

2.2.2. **Reduction of Order Method.** This method provides a way to obtain a second solution to a differential equation if we happen to know one solution.

Theorem 2.2.4 (Reduction of Order). *If a nonzero function y_1 is solution to*

$$y'' + p(t)\, y' + q(t)\, y = 0. \tag{2.2.1}$$

where p, q are given functions, then a second solution to this same equation is given by

$$y_2(t) = y_1(t) \int \frac{e^{-P(t)}}{y_1^2(t)}\, dt, \tag{2.2.2}$$

where $P(t) = \int p(t)\, dt$. Furthermore, y_1 and y_2 are fundamental solutions to Eq. (2.2.1).

Remark: In the first pat of the proof we write $y_2(t) = v(t)\, y_1(t)$ and show that y_2 is solution of Eq. (2.2.1) iff the function v is solution of

$$v'' + \Big(2\frac{y_1'(t)}{y_1(t)} + p(t)\Big)\, v' = 0. \tag{2.2.3}$$

In the second part we solve the equation for v. This is a first order equation for for $w = v'$, since v itself does not appear in the equation, hence the name reduction of order method. The equation for w is linear and first order, so we can solve it using the integrating factor method. Then one more integration gives the function v, which is the factor multiplying y_1 in Eq. (2.2.2).

Remark: The functions v and w in this subsection have no relation with the functions v and w from the previous subsection.

Proof of Theorem 2.2.4: We write $y_2 = v y_1$ and we put this function into the differential equation in 2.2.1, which give us an equation for v. To start, compute y_2' and y_2'',

$$y_2' = v'\, y_1 + v\, y_1', \qquad y_2'' = v''\, y_1 + 2v'\, y_1' + v\, y_1''.$$

Introduce these equations into the differential equation,

$$\begin{aligned} 0 &= (v''\, y_1 + 2v'\, y_1' + v\, y_1'') + p\, (v'\, y_1 + v\, y_1') + qv\, y_1 \\ &= y_1\, v'' + (2y_1' + p\, y_1)\, v' + (y_1'' + p\, y_1' + q\, y_1)\, v. \end{aligned}$$

The function y_1 is solution to the differential original differential equation,

$$y_1'' + p\, y_1' + q\, y_1 = 0,$$

then, the equation for v is given by

$$y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0. \quad \Rightarrow \quad v'' + \Big(2\frac{y_1'}{y_1} + p\Big)\, v' = 0.$$

This is Eq. (2.2.3). The function v does not appear explicitly in this equation, so denoting $w = v'$ we obtain

$$w' + \Big(2\frac{y_1'}{y_1} + p\Big)\, w = 0.$$

This is is a first order linear equation for w, so we solve it using the integrating factor method, with integrating factor

$$\mu(t) = y_1^2(t)\, e^{P(t)}, \quad \text{where} \quad P(t) = \int p(t)\, dt.$$

Therefore, the differential equation for w can be rewritten as a total t-derivative as

$$\big(y_1^2\, e^P\, w\big)' = 0 \quad \Rightarrow \quad y_1^2\, e^P\, w = w_0 \quad \Rightarrow \quad w(t) = w_0\, \frac{e^{-P(t)}}{y_1^2(t)}.$$

Since $v' = w$, we integrate one more time with respect to t to obtain

$$v(t) = w_0 \int \frac{e^{-P(t)}}{y_1^2(t)}\, dt + v_0.$$

We are looking for just one function v, so we choose the integration constants $w_0 = 1$ and $v_0 = 0$. We then obtain

$$v(t) = \int \frac{e^{-P(t)}}{y_1^2(t)}\, dt \quad \Rightarrow \quad y_2(t) = y_1(t) \int \frac{e^{-P(t)}}{y_1^2(t)}\, dt.$$

For the furthermore part, we now need to show that the functions y_1 and $y_2 = vy_1$ are linearly independent. We start computing their Wronskian,

$$W_{y_1 y_2} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & (v'y_1 + vy_1') \end{vmatrix} = y_1(v'y_1 + vy_1') - vy_1y_1' \quad \Rightarrow \quad W_{y_1y_2} = v'y_1^2.$$

Recall that above in this proof we have computed $v' = w$, and the result was $w = e^{-P}/y_1^2$. So we get $v'y_1^2 = e^{-P}$, and then the Wronskian is given by

$$W_{y_1y_2} = e^{-P}.$$

This is a nonzero function, therefore the functions y_1 and $y_2 = vy_1$ are linearly independent. This establishes the Theorem. □

Example 2.2.4: Find a second solution y_2 linearly independent to the solution $y_1(t) = t$ of the differential equation

$$t^2 y'' + 2ty' - 2y = 0.$$

Solution: We look for a solution of the form $y_2(t) = t\, v(t)$. This implies that

$$y_2' = t\, v' + v, \qquad y_2'' = t\, v'' + 2v'.$$

So, the equation for v is given by

$$\begin{aligned} 0 &= t^2\big(t\, v'' + 2v'\big) + 2t\big(t\, v' + v\big) - 2t\, v \\ &= t^3\, v'' + (2t^2 + 2t^2)\, v' + (2t - 2t)\, v \\ &= t^3\, v'' + (4t^2)\, v' \quad \Rightarrow \quad v'' + \frac{4}{t} v' = 0. \end{aligned}$$

Notice that this last equation is precisely Eq. (**??**), since in our case we have

$$y_1 = t, \qquad p(t) = \frac{2}{t} \quad \Rightarrow \quad t\, v'' + \big[2 + \frac{2}{t} t\big]\, v' = 0.$$

The equation for v is a first order equation for $w = v'$, given by

$$\frac{w'}{w} = -\frac{4}{t} \quad \Rightarrow \quad w(t) = c_1 t^{-4}, \qquad c_1 \in \mathbb{R}.$$

Therefore, integrating once again we obtain that

$$v = c_2 t^{-3} + c_3, \qquad c_2, c_3 \in \mathbb{R},$$

and recalling that $y_2 = t\,v$ we then conclude that

$$y_2 = c_2 t^{-2} + c_3 t.$$

Choosing $c_2 = 1$ and $c_3 = 0$ we obtain that $y_2(t) = t^{-2}$. Therefore, a fundamental solution set to the original differential equation is given by

$$y_1(t) = t, \qquad y_2(t) = \frac{1}{t^2}.$$

$\lhd$

2.2.3. Exercises.

2.2.1.- Find the solution y to the second order, nonlinear equation

$$t^2\, y'' + 6t\, y' = 1, \quad t > 0.$$

2.2.2.- .

2.3. Homogeneous Constant Coefficients Equations

All solutions to a second order linear homogeneous equation can be obtained from any pair of nonproportional solutions. This is the main notion in § 2.1, Theorem 2.1.7. In this section we obtain these two linearly independent solutions in the particular case that the equation has constant coefficients. Such problem reduces to solve for the roots of a degree two polynomial, the characteristic polynomial.

2.3.1. The Roots of the Characteristic Polynomial.

Thanks to the work done in § 2.1 we only need to find two linearly independent solutions to the second order linear homogeneous equation. Then Theorem 2.1.7 says that every other solution is a linear combination of the former two. How do we find any pair of linearly independent solutions? Since the equation is so simple, having constant coefficients, we find such solutions by trual amd error. Here is an example of this idea.

Example 2.3.1: Find solutions to the equation

$$y'' + 5y' + 6y = 0. \tag{2.3.1}$$

Solution: We try to find solutions to this equation using simple test functions. For example, it is clear that power functions $y = t^n$ won't work, since the equation

$$n(n-1)\, t^{(n-2)} + 5n\, t^{(n-1)} + 6\, t^n = 0$$

cannot be satisfied for all $t \in \mathbb{R}$. We obtained, instead, a condition on t. This rules out power functions. A key insight is to try with a test function having a derivative proportional to the original function, $y'(t) = r\, y(t)$. Such function would be simplified from the equation. For example, we try now with the test function $y(t) = e^{rt}$. If we introduce this function in the differential equation we get

$$(r^2 + 5r + 6)\, e^{rt} = 0 \quad \Leftrightarrow \quad r^2 + 5r + 6 = 0. \tag{2.3.2}$$

We have eliminated the exponential from the differential equation, and now the equation is a condition on the constant r. We now look for the appropriate values of r, which are the roots of a polynomial degree two,

$$r_{\pm} = \frac{1}{2}\big(-5 \pm \sqrt{25 - 24}\big) = \frac{1}{2}\,(-5 \pm 1) \quad \Rightarrow \quad \begin{cases} r_+ = -2, \\ r_- = -3. \end{cases}$$

We have obtained two different roots, which implies we have two different solutions,

$$y_1(t) = e^{-2t}, \qquad y_2(t) = e^{-3t}.$$

These solutions are not proportional to each other, so the are fundamental solutions to the differential equation in (2.3.1). Therefore, Theorem 2.1.7 in § 2.1 implies that we have found all possible solutions to the differential equation, and they are given by

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}. \tag{2.3.3}$$

$\lhd$

From the example above we see that this idea will produce fundamental solutions to all constant coefficients homogeneous equations having associated polynomials with two different roots. Such polynomial play an important role to find solutions to differential equations as the one above, so we give such polynomial a name.

Definition 2.3.1. *The* ***characteristic polynomial*** *and* ***characteristic equation*** *of the second order linear homogeneous equation with constant coefficients*

$$y'' + a_1 y' + a_0 = 0,$$

are given by

$$p(r) = r^2 + a_1 r + a_0, \qquad p(r) = 0.$$

As we saw in Example 2.3.1, the roots of the characteristic polynomial are crucial to express the solutions of the differential equation above. The characteristic polynomial is a second degree polynomial with real coefficients, and the general expression for its roots is

$$r_{\pm} = \frac{1}{2}\Big(-a_1 \pm \sqrt{a_1^2 - 4a_0}\Big).$$

If the discriminant $(a_1^2 - 4a_0)$ is positive, zero, or negative, then the roots of p are different real numbers, only one real number, or a complex-conjugate pair of complex numbers. For each case the solution of the differential equation can be expressed in different forms.

Theorem 2.3.2 (Constant Coefficients). *If $r_{\pm}$ are the roots of the characteristic polynomial to the second order linear homogeneous equation with constant coefficients*

$$y'' + a_1 y' + a_0 y = 0, \tag{2.3.4}$$

and if c_+, c_- are arbitrary constants, then the following statements hold true.

(a) If $r_+ \neq r_-$, real or complex, then the general solution of Eq. (2.3.4) is given by

$$y_{\rm gen}(t) = c_+ e^{r_+ t} + c_- e^{r_- t}.$$

(b) If $r_+ = r_- = r_0 \in \mathbb{R}$, then the general solution of Eq. (2.3.4) is given by

$$y_{\rm gen}(t) = c_+ e^{r_0 t} + c_- t e^{r_0 t}.$$

Furthermore, given real constants t_0, y_0 and y_1, there is a unique solution to the initial value problem given by Eq. (2.3.4) and the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$.

Remarks:

(a) The proof is to guess that functions $y(t) = e^{rt}$ must be solutions for appropriate values of the exponent constant r, the latter being roots of the characteristic polynomial. When the characteristic polynomial has two different roots, Theorem 2.1.7 says we have all solutions. When the root is repeated we use the reduction of order method to find a second solution not proportional to the first one.

(b) At the end of the section we show a proof where we construct the fundamental solutions y_1, y_2 without guessing them. We do not need to use Theorem 2.1.7 in this second proof, which is based completely in a generalization of the reduction of order method.

Proof of Theorem 2.3.2: We guess that particular solutions to Eq. 2.3.4 must be exponential functions of the form $y(t) = e^{rt}$, because the exponential will cancel out from the equation and only a condition for r will remain. This is what happens,

$$r^2 e^{rt} + a_1 e^{rt} + a_0 e^{rt} = 0 \quad \Rightarrow \quad r^2 + a_1 r + a_0 = 0.$$

The second equation says that the appropriate values of the exponent are the root of the characteristic polynomial. We now have two cases. If $r_+ \neq r_-$ then the solutions

$$y_+(t) = e^{r_+ t}, \quad y_-(t) = e^{r_- t},$$

are linearly independent, so the general solution to the differential equation is

$$y_{\rm gen}(t) = c_+ e^{r_+ t} + c_- e^{r_- t}.$$

If $r_+ = r_- = r_0$, then we have found only one solution $y_+(t) = e^{r_0 t}$, and we need to find a second solution not proportional to y_+. This is what the reduction of order method is perfect for. We write the second solution as

$$y_-(t) = v(t)\, y_+(t) \quad \Rightarrow \quad y_-(t) = v(t)\, e^{r_0 t},$$

and we put this expression in the differential equation (2.3.4),

$$\big(v'' + 2r_0 v' + v r_0^2\big)\, e^{r_0 t} + \big(v' + r_0 v\big)\, a_1 e^{r_0 t} + a_0 v\, e^{r_0 t} = 0.$$

We cancel the exponential out of the equation and we reorder terms,

$$v'' + (2r_0 + a_1)\, v' + (r_0^2 + a_1 r_0 + a_0)\, v = 0.$$

We now need to use that r_0 is a root of the characteristic polynomial, $r_0^2 + a_1 r_0 + a_0 = 0$, so the last term in the equation above vanishes. But we also need to use that the root r_0 is repeated,

$$r_0 = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0} = -\frac{a_1}{2} \quad \Rightarrow \quad 2r_0 + a_1 = 0.$$

The equation on the right side above implies that the second term in the differential equation for v vanishes. So we get that

$$v'' = 0 \quad \Rightarrow \quad v(t) = c_1 + c_2 t$$

and the second solution is $y_-(t) = (c_1 + c_2 t)\, y_+(t)$. If we choose the constant $c_2 = 0$, the function y_- is proportional to y_+. So we definitely want $c_2 \neq 0$. The other constant, c_1, only adds a term proportional to y_+, we can choose it zero. So the simplest choice is $c_1 = 0$, $c_2 = 1$, and we get the fundamental solutions

$$y_+(t) = e^{r_0 t}, \qquad y_-(t) = t\, e^{r_0 t}.$$

So the general solution for the repeated root case is

$$y_{\rm gen}(t) = c_+\, e^{r_0 t} + c_- t\, e^{r_0 t}.$$

The furthermore part follows from solving a 2×2 linear system for the unknowns c_+ and c_-. The initial conditions for the case $r_+ \neq r_-$ are the following,

$$y_0 = c_+\, e^{r_+ t_0} + c_-\, e^{r_- t_0}, \qquad y_1 = r_+ c_+\, e^{r_+ t_0} + r_- c_-\, e^{r_- t_0}.$$

It is not difficult to verify that this system is always solvable and the solutions are

$$c_+ = -\frac{(r_- y_0 - y_1)}{(r_+ - r_-)\, e^{r_+ t_0}}, \qquad c_- = \frac{(r_+ y_0 - y_1)}{(r_+ - r_-)\, e^{r_- t_0}}.$$

The initial conditions for the case $r_- = r_- = r_0$ are the following,

$$y_0 = (c_+ + c_- t_0)\, e^{r_0 t_0}, \qquad y_1 = c_-\, e^{r_0 t_0} + r_0 (c_+ + c_- t_0)\, e^{r_0 t_0}.$$

It is also not difficult to verify that this system is always solvable and the solutions are

$$c_+ = \frac{y_0 + t_0 (r_0 y_0 - y_1)}{e^{r_0 t_0}}, \qquad c_- = -\frac{(r_0 y_0 - y_0)}{e^{r_0 t_0}}.$$

This establishes the Theorem. □

EXAMPLE 2.3.2: Find the solution y of the initial value problem

$$y'' + 5y' + 6 = 0, \qquad y(0) = 1, \qquad y'(0) = -1.$$

SOLUTION: We know that the general solution of the differential equation above is

$$y_{\rm gen}(t) = c_+ e^{-2t} + c_- e^{-3t}.$$

We now find the constants c_+ and c_- that satisfy the initial conditions above,

$$\left.\begin{aligned} 1 = y(0) &= c_+ + c_- \\ -1 = y'(0) &= -2c_+ - 3c_- \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} c_+ = 2, \\ c_- = -1. \end{cases}$$

Therefore, the unique solution to the initial value problem is

$$y(t) = 2e^{-2t} - e^{-3t}.$$

$\lhd$

Example 2.3.3: Find the general solution y_{gen} of the differential equation

$$2y'' - 3y' + y = 0.$$

Solution: We look for every solutions of the form $y(t) = e^{rt}$, where r is solution of the characteristic equation

$$2r^2 - 3r + 1 = 0 \quad \Rightarrow \quad r = \frac{1}{4}\big(3 \pm \sqrt{9-8}\big) \quad \Rightarrow \quad \begin{cases} r_+ = 1, \\ r_- = \dfrac{1}{2}. \end{cases}$$

Therefore, the general solution of the equation above is

$$y_{\text{gen}}(t) = c_+ e^{t} + c_- e^{t/2}.$$

$\lhd$

Example 2.3.4: Find the general solution y_{gen} of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: We first find the roots of the characteristic polynomial,

$$r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(2 \pm \sqrt{4-24}\big) \quad \Rightarrow \quad r_\pm = 1 \pm i\sqrt{5}.$$

Since the roots of the characteristic polnomial are different, Theorem 2.3.2 says that the general solution of the differential equation above, which includes complex-valued solutions, can be written as follows,

$$y_{\text{gen}}(t) = \tilde{c}_+ \, e^{(1+i\sqrt{5})t} + \tilde{c}_- \, e^{(1-i\sqrt{5})t}, \qquad \tilde{c}_+, \tilde{c}_- \in \mathbb{C}.$$

$\lhd$

Example 2.3.5: Find the solution to the initial value problem

$$9y'' + 6y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = \frac{5}{3}.$$

Solution: The characteristic polynomial is $p(r) = 9r^2 + 6r + 1$, with roots given by

$$r_\pm = \frac{1}{18}\big(-6 \pm \sqrt{36-36}\big) \quad \Rightarrow \quad r_+ = r_- = -\frac{1}{3}.$$

Theorem 2.3.2 says that the general solution has the form

$$y_{\text{gen}}(t) = c_+ \, e^{-t/3} + c_- t \, e^{-t/3}.$$

We need to compute the derivative of the expression above to impose the initial conditions,

$$y'_{\text{gen}}(t) = -\frac{c_+}{3} \, e^{-t/3} + c_- \Big(1 - \frac{t}{3}\Big) \, e^{-t/3},$$

then, the initial conditions imply that

$$\left.\begin{aligned} 1 &= y(0) = c_+, \\ \frac{5}{3} &= y'(0) = -\frac{c_+}{3} + c_- \end{aligned}\right\} \quad \Rightarrow \quad c_+ = 1, \qquad c_- = 2.$$

So, the solution to the initial value problem above is: $y(t) = (1 + 2t)\, e^{-t/3}$. ◁

2.3.2. **Real Solutions for Complex Roots.** We study in more detail the solutions to the differential equation (2.3.4) in the case that the characteristic polynomial has complex roots. Since these roots have the form

$$r_\pm = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0},$$

the roots are complex-valued in the case $a_1^2 - 4a_0 < 0$. We use the notation

$$r_\pm = \alpha \pm i\beta, \quad \text{with} \quad \alpha = -\frac{a_1}{2}, \quad \beta = \sqrt{a_0 - \frac{a_1^2}{4}}.$$

The fundamental solutions in Theorem 2.3.2 are the complex-valued functions

$$\tilde{y}_+ = e^{(\alpha + i\beta)t}, \qquad \tilde{y}_- = e^{(\alpha - i\beta)t}.$$

The general solution constructed from these solutions is

$$y_{\text{gen}}(t) = \tilde{c}_+\, e^{(\alpha+i\beta)t} + \tilde{c}_-\, e^{(\alpha-i\beta)t}, \qquad \tilde{c}_+,\ \tilde{c}_- \in \mathbb{C}.$$

This formula for the general solution includes real valued and complex valued solutions. But it is not so simple to single out the real valued solutions. Knowing the real valued solutions could be important in physical applications. If a physical system is described by a differential equation with real coefficients, more often than not one is interested in finding real valued solutions. For that reason we now provide a new set of fundamental solutions that are real valued. Using real valued fundamental solution is simple to separate all real valued solutions from the complex valued ones.

Theorem 2.3.3 (Real Valued Fundamental Solutions). *If the differential equation*

$$y'' + a_1\, y' + a_0\, y = 0, \tag{2.3.5}$$

where a_1, a_0 are real constants, has characteristic polynomial with complex roots $r_\pm = \alpha \pm i\beta$ and complex valued fundamental solutions

$$\tilde{y}_+(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_-(t) = e^{(\alpha-i\beta)t},$$

then the equation also has real valued fundamental solutions given by

$$y_+(t) = e^{\alpha t}\cos(\beta t), \qquad y_-(t) = e^{\alpha t}\sin(\beta t).$$

Proof of Theorem 2.3.3: We start with the complex valued fundamental solutions

$$\tilde{y}_+(t) = e^{(\alpha+i\beta)t}, \qquad \tilde{y}_-(t) = e^{(\alpha-i\beta)t}.$$

We take the function $\tilde{y}_+$ and we use a property of complex exponentials,

$$\tilde{y}_+(t) = e^{(\alpha+i\beta)t} = e^{\alpha t}\, e^{i\beta t} = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big),$$

where on the last step we used Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Repeat this calculation for y_- we get,

$$\tilde{y}_+(t) = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big), \quad \tilde{y}_-(t) = e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big).$$

If we recall the superposition property of linear homogeneous equations, Theorem 2.1.5, we know that any linear combination of the two solutions above is also a solution of the differential equation (2.3.6), in particular the combinations

$$y_+(t) = \frac{1}{2}\big(\tilde{y}_+(t) + \tilde{y}_-(t)\big), \quad y_-(t) = \frac{1}{2i}\big(\tilde{y}_+(t) - \tilde{y}_-(t)\big).$$

A straightforward computation gives

$$y_+(t) = e^{\alpha t}\cos(\beta t), \qquad y_-(t) = e^{\alpha t}\sin(\beta t).$$

This establishes the Theorem. □

Example 2.3.6: Find the real valued general solution of the equation

$$y'' - 2y' + 6y = 0.$$

Solution: We already found the roots of the characteristic polynomial, but we do it again,

$$r^2 - 2r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(2 \pm \sqrt{4-24}\big) \quad \Rightarrow \quad r_\pm = 1 \pm i\sqrt{5}.$$

So the complex valued fundamental solutions are

$$\tilde{y}_+(t) = e^{(1+i\sqrt{5})\,t}, \quad \tilde{y}_-(t) = e^{(1-i\sqrt{5})\,t}.$$

Theorem **??** says that real valued fundamental solutions are given by

$$y_+(t) = e^t\cos(\sqrt{5}t), \quad y_-(t) = e^t\sin(\sqrt{5}t).$$

So the real valued general solution is given by

$$y_{\text{gen}}(t) = \big(c_+\cos(\sqrt{5}\,t) + c_-\sin(\sqrt{5}\,t)\big)\,e^t, \quad c_+,\ c_- \in \mathbb{R}.$$

◁

Remark: Sometimes it is difficult to remember the formula for real valued solutions. One way to obtain those solutions without remembering the formula is to start repeat the proof of Theorem 2.3.3. Start with the complex valued solution $\tilde{y}_+$ and use the properties of the complex exponential,

$$\tilde{y}_+(t) = e^{(1+i\sqrt{5})t} = e^t\,e^{i\sqrt{5}t} = e^t\,\big(\cos(\sqrt{5}t) + i\sin(\sqrt{5}t)\big).$$

The real valued fundamental solutions are the real and imaginary parts in that expression.

Example 2.3.7: Find real valued fundamental solutions to the equation

$$y'' + 2\,y' + 6\,y = 0.$$

Solution: The roots of the characteristic polynomial $p(r) = r^2 + 2r + 6$ are

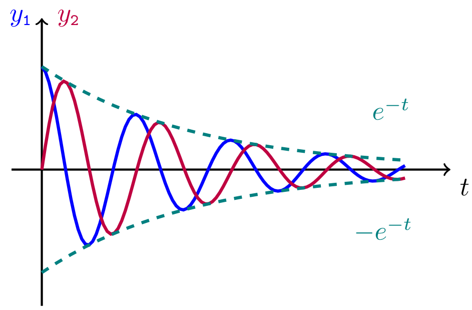
$$r_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4-24}\big] = \frac{1}{2}\big[-2 \pm \sqrt{-20}\big] \quad \Rightarrow \quad r_\pm = -1 \pm i\sqrt{5}.$$

These are complex-valued roots, with

$$\alpha = -1, \qquad \beta = \sqrt{5}.$$

Real-valued fundamental solutions are

$$y_1(t) = e^{-t}\,\cos(\sqrt{5}\,t), \qquad y_2(t) = e^{-t}\,\sin(\sqrt{5}\,t).$$



Second order differential equations with characteristic polynomials having complex roots, like the one in this example, describe physical processes related to damped oscillations. An example from physics is a pendulums with friction. ◁

FIGURE 10. Solutions from Ex. 2.3.7.

EXAMPLE 2.3.8: Find the real valued general solution of $y'' + 5\,y = 0$.

SOLUTION: The characteristic polynomial is $p(r) = r^2 + 5$, with roots $r_\pm = \pm\sqrt{5}\,i$. In this case $\alpha = 0$, and $\beta = \sqrt{5}$. Real valued fundamental solutions are

$$y_+(t) = \cos(\sqrt{5}\,t), \qquad y_-(t) = \sin(\sqrt{5}\,t).$$

The real valued general solution is

$$y_{\rm gen}(t) = c_+ \cos(\sqrt{5}\,t) + c_- \sin(\sqrt{5}\,t), \qquad c_+,\, c_- \in \mathbb{R}.$$ ◁

Remark: Physical processes that oscillate in time without dissipation could be described by differential equations like the one in this example.

2.3.3. **Constructive proof of Theorem 2.3.2.** We now present an alternative proof for Theorem 2.3.2 that does not involve guessing the fundamental solutions of the equation. Instead, we construct these solutions using a generalization of the reduction of order method.

Proof of Theorem 2.3.2: The proof has two main parts: First, we transform the original equation into an equation simpler to solve for a new unknown; second, we solve this simpler problem.

In order to transform the problem into a simpler one, we express the solution y as a product of two functions, that is, $y(t) = u(t)v(t)$. Choosing v in an appropriate way the equation for u will be simpler to solve than the equation for y. Hence,

$$y = uv \quad \Rightarrow \quad y' = u'v + v'u \quad \Rightarrow \quad y'' = u''v + 2u'v' + v''u.$$

Therefore, Eq. (2.3.4) implies that

$$(u''v + 2u'v' + v''u) + a_1\,(u'v + v'u) + a_0\,uv = 0,$$

that is,

$$\Big[u'' + \Big(a_1 + 2\frac{v'}{v}\Big)\,u' + a_0\,u\Big]v + (v'' + a_1\,v')\,u = 0. \tag{2.3.6}$$

We now choose the function v such that

$$a_1 + 2\frac{v'}{v} = 0 \quad \Leftrightarrow \quad \frac{v'}{v} = -\frac{a_1}{2}. \tag{2.3.7}$$

We choose a simple solution of this equation, given by

$$v(t) = e^{-a_1 t/2}.$$

Having this expression for v one can compute v' and v'', and it is simple to check that

$$v'' + a_1 v' = -\frac{a_1^2}{4}\,v. \tag{2.3.8}$$

Introducing the first equation in (2.3.7) and Eq. (2.3.8) into Eq. (2.3.6), and recalling that v is non-zero, we obtain the simplified equation for the function u, given by

$$u'' - k\,u = 0, \qquad k = \frac{a_1^2}{4} - a_0. \tag{2.3.9}$$

Eq. (2.3.9) for u is simpler than the original equation (2.3.4) for y since in the former there is no term with the first derivative of the unknown function.

In order to solve Eq. (2.3.9) we repeat the idea followed to obtain this equation, that is, express function u as a product of two functions, and solve a simple problem of one of the functions. We first consider the harder case, which is when $k \neq 0$. In this case, let us express $u(t) = e^{\sqrt{k}t}\,w(t)$. Hence,

$$u' = \sqrt{k}e^{\sqrt{k}t}\,w + e^{\sqrt{k}t}\,w' \quad \Rightarrow \quad u'' = ke^{\sqrt{k}t}\,w + 2\sqrt{k}e^{\sqrt{k}t}\,w' + e^{\sqrt{k}t}\,w''.$$

Therefore, Eq. (2.3.9) for function u implies the following equation for function w

$$0 = u'' - ku = e^{\sqrt{k}t}\,(2\sqrt{k}\,w' + w'') \quad \Rightarrow \quad w'' + 2\sqrt{k}w' = 0.$$

Only derivatives of w appear in the latter equation, so denoting $x(t) = w'(t)$ we have to solve a simple equation

$$x' = -2\sqrt{k}\,x \quad \Rightarrow \quad x(t) = x_0 e^{-2\sqrt{k}t}, \qquad x_0 \in \mathbb{R}.$$

Integrating we obtain w as follows,

$$w' = x_0 e^{-2\sqrt{k}t} \quad \Rightarrow \quad w(t) = -\frac{x_0}{2\sqrt{k}} e^{-2\sqrt{k}t} + c_0.$$

renaming $c_1 = -x_0/(2\sqrt{k})$, we obtain

$$w(t) = c_1 e^{-2\sqrt{k}t} + c_0 \quad \Rightarrow \quad u(t) = c_0 e^{\sqrt{k}t} + c_1 e^{-\sqrt{k}t}.$$

We then obtain the expression for the solution $y = uv$, given by

$$y(t) = c_0 e^{(-\frac{a_1}{2}+\sqrt{k})t} + c_1 e^{(-\frac{a_1}{2}-\sqrt{k})t}.$$

Since $k = (a_1^2/4 - a_0)$, the numbers

$$r_{\pm} = -\frac{a_1}{2} \pm \sqrt{k} \quad \Leftrightarrow \quad r_{\pm} = \frac{1}{2}\Big(-a_1 \pm \sqrt{a_1^2 - 4a_0}\Big)$$

are the roots of the characteristic polynomial

$$r^2 + a_1\,r + a_0 = 0,$$

we can express all solutions of the Eq. (2.3.4) as follows

$$y(t) = c_0 e^{r_+ t} + c_1 e^{r_- t}, \qquad k \neq 0.$$

Finally, consider the case $k = 0$. Then, Eq. (2.3.9) is simply given by

$$u'' = 0 \quad \Rightarrow \quad u(t) = (c_0 + c_1 t) \qquad c_0, c_1 \in \mathbb{R}.$$

Then, the solution y to Eq. (2.3.4) in this case is given by

$$y(t) = (c_0 + c_1 t)\,e^{-a_1 t/2}.$$

Since $k = 0$, the characteristic equation $r^2 + a_1\,r + a_0 = 0$ has only one root $r_+ = r_- = -a_1/2$, so the solution y above can be expressed as

$$y(t) = (c_0 + c_1 t)\,e^{r_+ t}, \qquad k = 0.$$

The Furthermore part is the same as in Theorem 2.3.2. This establishes the Theorem. $\square$

Notes.

In the case that the characteristic polynomial of a differential equation has repeated roots there is an interesting argument to guess the solution y_-. The idea is to take a particular type of limit in solutions of differential equations with complex valued roots.

Consider the equation in (2.3.4) with a characteristic polynomial having complex valued roots given by $r_\pm = \alpha \pm i\beta$, with

$$\alpha = -\frac{a_1}{2}, \qquad \beta = \sqrt{a_0 - \frac{a_1^2}{4}}.$$

Real valued fundamental solutions in this case are given by

$$\hat{y}_+ = e^{\alpha\, t}\, \cos(\beta t), \quad \hat{y}_- = e^{\alpha\, t}\, \sin(\beta t).$$

We now study what happen to these solutions $\hat{y}_+$ and $\hat{y}_-$ in the following limit: The variable t is held constant, α is held constant, and $\beta \to 0$. The last two conditions are conditions on the equation coefficients, a_1, a_0. For example, we fix a_1 and we vary $a_0 \to a_1^2/4$ from above.

Since $\cos(\beta t) \to 1$ as $\beta \to 0$ with t fixed, then keeping α fixed too, we obtain

$$\hat{y}_+(t) = e^{\alpha\, t} \cos(\beta t) \;\longrightarrow\; e^{\alpha\, t} = y_+(t).$$

Since $\dfrac{\sin(\beta t)}{\beta t} \to 1$ as $\beta \to 0$ with t constant, that is, $\sin(\beta t) \to \beta t$, we conclude that

$$\frac{\hat{y}_-(t)}{\beta} = \frac{\sin(\beta t)}{\beta}\, e^{\alpha\, t} = \frac{\sin(\beta t)}{\beta t}\, t\, e^{\alpha\, t} \;\longrightarrow\; t\, e^{\alpha\, t} = y_-(t).$$

The calculation above says that the function $\hat{y}_-/\beta$ is close to the function $y_-(t) = t\, e^{\alpha\, t}$ in the limit $\beta \to 0$, t held constant. This calculation provides a candidate, $y_-(t) = t\, y_+(t)$, of a solution to Eq. (2.3.4). It is simple to verify that this candidate is in fact solution of Eq. (2.3.4). Since y_- is not proportional to y_+, one then concludes the functions y_+, y_- are a fundamental set for the differential equation in (2.3.4) in the case the characteristic polynomial has repeated roots.

2.3.4. Exercises.

2.3.1.- .

2.3.2.- .

2.4. Nonhomogeneous Equations

All solutions of a linear homogeneous equation can be obtained from only two solutions that are linearly independent, called fundamental solutions. Every other solution is a linear combination of these two. This is the general solution formula for homogeneous equations, and it is the main result in § 2.1, Theorem 2.1.7. This result is not longer true for nonhomogeneous equations. The superposition property, Theorem 2.1.5, which played an important part to get the general solution formula for homogeneous equations, is not true for nonhomogeneous equations. In this section we prove a new general solution formula that is true for nonhomogeneous equations. We show that all solutions of a linear nonhomogeneous equation can be obtained from only three functions. The first two functions are fundamental solutions of the homogeneous equation. The third function is one single solution of the nonhomogeneous equation. It does not matter which one. It is called a particular solution of the nonhomogeneous equation. Then every other solution of the nonhomogeneous equation is obtained from these three functions.

In this section we show two different ways to compute the particular solution of a nonhomogeneous equation, the undetermined coefficients method and the variation of parameters method. In the former method we guess a particular solution from the expression of the source in the equation. The guess contains a few unknown constants, the undetermined coefficients, that must be determined by the equation. The undetermined method works for constant coefficients linear operators and simple source functions. The source functions and the associated guessed solutions are collected in a small table. This table is constructed by trial and error, and the calculation to find the coefficients in the solutions are simple. In the latter method we have a formula to compute a particular solution in terms of the equation source, and the fundamental solutions of the homogeneous equation. The variation of parameters method works with variable coefficients linear operators and general source functions. But the calculations to find the solution are usually not so simple as in the undetermined coefficients method.

2.4.1. The General Solution Formula.

The general solution formula for homogeneous equations, Theorem 2.1.7, is no longer true for nonhomogeneous equations. But there is a general solution formula for nonhomogeneous equations. Such formula involves three functions, two of them are fundamental solutions of the homogeneous equation, and the third function is any solution of the nonhomogeneous equation. Every other solution of the nonhomogeneous equation can be obtained from these three functions.

Theorem 2.4.1 (General Solution). *Every solution y of the nonhomogeneous equation*

$$L(y) = f, \tag{2.4.1}$$

with $L(y) = y'' + p\,y' + q\,y$, where p, q, and f are continuous functions, is given by

$$y = c_1\,y_1 + c_2\,y_2 + y_p,$$

where the functions y_1 and y_2 are fundamental solutions of the homogeneous equation, $L(y_1) = 0$, $L(y_2) = 0$, and y_p is any solution of the nonhomogeneous equation $L(y_p) = f$.

Before we proof Theorem 2.4.1 we state the following definition, which comes naturally from this Theorem.

Definition 2.4.2. *The **general solution** of the nonhomogeneous equation $L(y) = f$ is a two-parameter family of functions*

$$y_{\rm gen}(t) = c_1\,y_1(t) + c_2\,y_2(t) + y_p(t), \tag{2.4.2}$$

where the functions y_1 and y_2 are fundamental solutions of the homogeneous equation, $L(y_1) = 0$, $L(y_2) = 0$, and y_p is any solution of the nonhomogeneous equation $L(y_p) = f$.

Remark: The difference of any two solutions of the nonhomogeneous equation is actually a solution of the homogeneous equation. This is the key idea to prove Theorem 2.4.1.

Proof of Theorem 2.4.1: Let y be any solution of the nonhomogeneous equation $L(y) = f$. Recall that we already have one solution, y_p, of the nonhomogeneous equation, $L(y_p) = f$. We can now subtract the second equation from the first,

$$L(y) - L(y_p) = f - f = 0 \quad \Rightarrow \quad L(y - y_p) = 0.$$

The equation on the right is obtained from the linearity of the operator L. This last equation says that the difference of any two solutions of the nonhomogeneous equation is solution of the homogeneous equation. The general solution formula for homogeneous equations says that all solutions of the homogeneous equation can be written as linear combinations of a pair of fundamental solutions, y_1, y_2. So the exist constants c_1, c_2 such that

$$y - y_p = c_1\, y_1 + c_2\, y_2.$$

Since for every y solution of $L(y) = f$ we can find constants c_1, c_2 such that the equation above holds true, we have found a formula for all solutions of the nonhomogeneous equation. This establishes the Theorem. □

2.4.2. **The Undetermined Coefficients Method.** The general solution formula in (2.4.2) is the most useful if there is a way to find a particular solution y_p of the nonhomogeneous equation $L(y_p) = f$. We now present a method to find such particular solution, the Undetermined Coefficients Method. This method works for *linear operators L with constant coefficients* and for *simple source functions f*. Here is a summary of the Undetermined Coefficients Method:

(1) Find fundamental solutions y_1, y_2 of the homogeneous equation $L(y) = 0$.
(2) Given the source functions f, guess the solutions y_p following the Table 1 below.
(3) If the function y_p given by the table satisfies $L(y_p) = 0$, then change the guess to $ty_{p.}$. If ty_p satisfies $L(ty_p) = 0$ as well, then change the guess to $t^2 y_p$.
(4) Find the undetermined constants k in the function y_p using the equation $L(y_p) = f$.

$f(t)$ (Source) (K, m, a, b, given.)	$y_p(t)$ (Guess) (k not given.)
Ke^{at}	ke^{at}
$K_m t^m + \cdots + K_0$	$k_m t^m + \cdots + k_0$
$K_1 \cos(bt) + K_2 \sin(bt)$	$k_1 \cos(bt) + k_2 \sin(bt)$
$(K_m t^m + \cdots + K_0)\, e^{at}$	$(k_m t^m + \cdots + k_0)\, e^{at}$
$\big(K_1 \cos(bt) + K_2 \sin(bt)\big)\, e^{at}$	$\big(k_1 \cos(bt) + k_2 \sin(bt)\big)\, e^{at}$
$(K_m t^m + \cdots + K_0)\big(\tilde{K}_1 \cos(bt) + \tilde{K}_2 \sin(bt)\big)$	$(k_m t^m + \cdots + k_0)\big(\tilde{k}_1 \cos(bt) + \tilde{k}_2 \sin(bt)\big)$

TABLE 1. List of sources f and solutions y_p to the equation $L(y_p) = f$.

This is the undetermined coefficients method. It is a set of simple rules to find a particular solution y_p of an nonhomogeneous equation $L(y_p) = f$ in the case that the source function f is one of the entries in the Table 1. There are a few formulas in particular cases and a few generalizations of the whole method. We discuss them after a few examples.

EXAMPLE 2.4.1: Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

SOLUTION: From the problem we get $L(y) = y'' - 3y' - 4y$ and $f(t) = 3e^{2t}$.

(1): Find fundamental solutions y_+, y_- to the homogeneous equation $L(y) = 0$. Since the homogeneous equation has constant coefficients we find the characteristic equation

$$r^2 - 3r - 4 = 0 \quad \Rightarrow \quad r_+ = 4, \quad r_- = -1, \quad \Rightarrow \quad y_{tipl}(t) = e^{4t}, \quad y_- = (t) = e^{-t}.$$

(2): The table says: For $f(t) = 3e^{2t}$ guess $y_p(t) = k\, e^{2t}$. The constant k is the undetermined coefficient we must find.

(3): Since $y_p(t) = k\, e^{2t}$ is not solution of the homogeneous equation, we do not need to modify our guess. (Recall: $L(y) = 0$ iff exist constants c_+, c_- such that $y(t) = c_+\, e^{4t} + c_-\, e^{-t}$.)

(4): Introduce y_p into $L(y_p) = f$ and find k. So we do that,

$$(2^2 - 6 - 4)ke^{2t} = 3e^{2t} \quad \Rightarrow \quad -6k = 3 \quad \Rightarrow \quad k = -\frac{1}{2}.$$

We guessed that y_p must be proportional to the exponential e^{2t} in order to cancel out the exponentials in the equation above. We have obtained that

$$y_p(t) = -\frac{1}{2}\, e^{2t}.$$

The undetermined coefficients method gives us a way to compute a particular solution y_p of the nonhomogeneous equation. We now use the general solution theorem, Theorem 2.4.1, to write the general solution of the nonhomogeneous equation,

$$y_{\text{gen}}(t) = c_+\, e^{4t} + c_-\, e^{-t} - \frac{1}{2}\, e^{2t}.$$

◁

Remark: The step (4) in Example 2.4.1 is a particular case of the following statement.

Lemma 2.4.3. *Consider a nonhomogeneous equation $L(y) = f$ with a constant coefficient operator L and characteristic polynomial p. If the source function is $f(t) = K\, e^{at}$, with $p(a) \neq 0$, then a particular solution of the nonhomogeneous equation is*

$$y_p(t) = \frac{K}{p(a)}\, e^{at}.$$

Proof of Lemma 2.4.3: Since the linear operator L has constant coefficients, let us write L and its associated characteristic polynomial p as follows,

$$L(y) = y'' + a_1 y' + a_0 y, \qquad p(r) = r^2 + a_1 r + a_0.$$

Since the source function is $f(t) = K\, e^{at}$, the Table 1 says that a good guess for a particular soution of the nonhomogneous equation is $y_p(t) = k\, e^{at}$. Our hypothesis is that this guess is not solution of the homogenoeus equation, since

$$L(y_p) = (a^2 + a_1 a + a_0)\, k\, e^{at} = p(a)\, k\, e^{at}, \quad \text{and} \quad p(a) \neq 0.$$

We then compute the constant k using the equation $L(y_p) = f$,

$$(a^2 + a_1 a + a_0)\, k\, e^{at} = K\, e^{at} \quad \Rightarrow \quad p(a)\, k\, e^{at} = K\, e^{at} \quad \Rightarrow \quad k = \frac{K}{p(a)}.$$

We get the particular solution $y_p(t) = \dfrac{K}{p(a)}\, e^{at}$. This establishes the Lemma. □

Remark: As we said, the step (4) in Example 2.4.1 is a particular case of Lemma 2.4.3,

$$y_p(t) = \frac{3}{p(2)}\, e^{2t} = \frac{3}{(2^2 - 6 - 4)}\, e^{2t} = \frac{3}{-6}\, e^{2t} \quad \Rightarrow \quad y_p(t) = -\frac{1}{2}\, e^{2t}.$$

In the following example our first guess for a particular solution y_p happens to be a solution of the homogenous equation.

EXAMPLE 2.4.2: Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\, e^{4t}.$$

SOLUTION: If we write the equation as $L(y) = f$, with $f(t) = 3\, e^{4t}$, then the operator L is the same as in Example 2.4.1. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in that example,

$$y_{+}(t) = e^{4t}, \qquad y_{-}(t) = e^{-t}.$$

The source function is $f(t) = 3\, e^{4t}$, so the Table 1 says that we need to guess $y_p(t) = k\, e^{4t}$. However, this function y_p is solution of the homogeneous equation, because

$$y_p = k\, y_{+}.$$

We have to change our guess, as indicated in the undetermined coefficients method, step (3)

$$y_p(t) = kt\, e^{4t}.$$

This new guess is not solution of the homogeneous equation. So we proceed to compute the constant k. We introduce the guess into $L(y_p) = f$,

$$y_p' = (1 + 4t)\, k\, e^{4t}, \qquad y_p'' = (8 + 16t)\, k\, e^{4t} \quad \Rightarrow \quad \big[8 - 3 + (16 - 12 - 4)t\big]\, k\, e^{4t} = 3\, e^{4t},$$

therefore, we get that

$$5k = 3 \quad \Rightarrow \quad k = \frac{3}{5} \quad \Rightarrow \quad y_p(t) = \frac{3}{5}\, t\, e^{4t}.$$

The general solution theorem for nonhomogneneous equations says that

$$y_{\text{gen}}(t) = c_{+}\, e^{4t} + c_{-}\, e^{-t} + \frac{3}{5}\, t\, e^{4t}.$$

$\lhd$

In the following example the equation source is a trigonometric function.

EXAMPLE 2.4.3: Find all the solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

SOLUTION: If we write the equation as $L(y) = f$, with $f(t) = 2\sin(t)$, then the operator L is the same as in Example 2.4.1. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in that example,

$$y_{+}(t) = e^{4t}, \qquad y_{-}(t) = e^{-t}.$$

Since the source function is $f(t) = 2\sin(t)$, the Table 1 says that we need to choose the function $y_p(t) = k_1 \cos(t) + k_2 \sin(t)$. This function y_p is not solution to the homogeneous equation. So we look for the constants k_1, k_2 using the differential equation,

$$y_p' = -k_1 \sin(t) + k_2 \cos(t), \qquad y_p'' = -k_1 \cos(t) - k_2 \sin(t),$$

and then we obtain

$$[-k_1 \cos(t) - k_2 \sin(t)] - 3[-k_1 \sin(t) + k_2 \cos(t)] - 4[k_1 \cos(t) + k_2 \sin(t)] = 2\sin(t).$$

Reordering terms in the expression above we get

$$(-5k_1 - 3k_2)\cos(t) + (3k_1 - 5k_2)\sin(t) = 2\sin(t).$$

The last equation must hold for all $t \in \mathbb{R}$. In particular, it must hold for $t = \pi/2$ and for $t = 0$. At these two points we obtain, respectively,

$$\left.\begin{aligned} 3k_1 - 5k_2 &= 2, \\ -5k_1 - 3k_2 &= 0, \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} k_1 = \dfrac{3}{17}, \\ k_2 = -\dfrac{5}{17}. \end{cases}$$

So the particular solution to the nonhomogeneous equation is given by

$$y_p(t) = \frac{1}{17}\big[3\cos(t) - 5\sin(t)\big].$$

The general solution theorem for nonhomogeneous equations implies

$$y_{\text{gen}}(t) = c_+ e^{4t} + c_- e^{-t} + \frac{1}{17}\big[3\cos(t) - 5\sin(t)\big].$$

$\lhd$

The next example collects a few nonhomogeneous equations and the guessed function y_p.

Example 2.4.4: We provide few more examples of nonhomogeneous equations and the appropriate guesses for the particular solutions.

(a) For $y'' - 3y' - 4y = 3e^{2t}\sin(t)$, guess, $y_p(t) = \big[k_1\cos(t) + k_2\sin(t)\big]\,e^{2t}$.

(b) For $y'' - 3y' - 4y = 2t^2\,e^{3t}$, guess, $y_p(t) = \big(k_2 t^2 + k_1 t + k_0\big)\,e^{3t}$.

(c) For $y'' - 3y' - 4y = 2t^2\,e^{4t}$, guess, $y_p(t) = \big(k_2 t^2 + k_1 t + k_0\big)\,t\,e^{4t}$.

(d) For $y'' - 3y' - 4y = 3t\,\sin(t)$, guess, $y_p(t) = (k_1 t + k_0)\,\big[\tilde{k}_1\cos(t) + \tilde{k}_2\sin(t)\big]$.

$\lhd$

Remark: Suppose that the source function f does not appear in Table 1, but f can be written as $f = f_1 + f_2$, with f_1 and f_2 in the table. In such case look for a particular solution $y_p = y_{p_1} + y_{p_2}$, where $L(y_{p_1}) = f_1$ and $L(y_{p_2}) = f_2$. Since the operator L is linear,

$$L(y_p) = L(y_{p_1} + y_{p_2}) = L(y_{p_1}) + L(y_{p_2}) = f_1 + f_2 = f \quad \Rightarrow \quad L(y_p) = f.$$

Example 2.4.5: Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\,e^{2t} + 2\sin(t).$$

Solution: If we write the equation as $L(y) = f$, with $f(t) = 2\sin(t)$, then the operator L is the same as in Example 2.4.1 and 2.4.3. So the solutions of the homogeneous equation $L(y) = 0$, are the same as in these examples,

$$y_+(t) = e^{4t}, \qquad y_-(t) = e^{-t}.$$

The source function $f(t) = 3\,e^{2t} + 2\sin(t)$ does not appear in Table 1, but each term does, $f_1(t) = 3\,e^{2t}$ and $f_2(t) = 2\sin(t)$. So we look for a particular solution of the form

$$y_p = y_{p_1} + y_{p_2}, \quad \text{where} \quad L(y_{p_1}) = 3\,e^{2t}, \quad L(y_{p_2}) = 2\sin(t).$$

We have chosen this example because we have solved each one of these equations before, in Example 2.4.1 and 2.4.3. We found the solutions

$$y_{p_1}(t) = -\frac{1}{2}\,e^{2t}, \qquad y_{p_2}(t) = \frac{1}{17}\big(3\cos(t) - 5\sin(t)\big).$$

Therefore, the particular solution for the equation in this example is

$$y_p(t) = -\frac{1}{2}\, e^{2t} + \frac{1}{17}\, \big(3\cos(t) - 5\sin(t)\big).$$

Using the general solution theorem for nonhomogeneous equations we obtain

$$y_{\text{gen}}(t) = c_+\, e^{4t} + c_-\, e^{-t} - \frac{1}{2}\, e^{2t} + \frac{1}{17}\, \big(3\cos(t) - 5\sin(t)\big).$$

◁

2.4.3. **The Variation of Parameters Method.** This method provides a second way to find a particular solution y_p to a nonhomogeneous equation $L(y) = f$. We summarize this method in formula to compute y_p in terms of any pair of fundamental solutions to the homogeneous equation $L(y) = 0$. The variation of parameters method works with second order linear equations having *variable coefficients* and contiuous but otherwise *arbitrary sources*. When the source function of a nonhomogeneous equation is simple enough to appear in Table 1 the undetermined coefficients method is a quick way to find a particular solution to the equation. When the source is more complicated, one usually turns to the variation of parameters method, with its more involved formula for a particular solution.

Theorem 2.4.4 (Variation of Parameters). *A particular solution to the equation*

$$L(y) = f,$$

with $L(y) = y'' + p(t)\, y' + q(t)\, y$ *and* p, q, f *continuous functions, is given by*

$$y_p = u_1 y_1 + u_2 y_2,$$

where y_1, y_2 *are fundamental solutions of the homogeneous equatio* $L(y) = 0$ *and the functions* u_1, u_2 *are defined by*

$$u_1(t) = \int -\frac{y_2(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t) f(t)}{W_{y_1 y_2}(t)}\, dt, \tag{2.4.3}$$

where $W_{y_1 y_2}$ *is the Wronskian of* y_1 *and* y_2.

The proof rests in a generalization of the reduction order method. Recall that the reduction order method is a way to find a second solution y_2 of an homogeneous equation if we already know one solution y_1. One writes $y_2 = u\, y_1$ and the original equation $L(y_2) = 0$ provides an equation for u. This equation for u is simpler than the original equation for y_2 because the function y_1 satisfies $L(y_1) = 0$.

The formula for y_p is obtained generalizing the reduction order method. We write y_p in terms of both fundamental solutions y_1, y_2 of the homogeneous equation,

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t).$$

We put this y_p in the equation $L(y_p) = f$ and we find an equation relating u_1 and u_2. It is important to realize that we have added one new function to the original problem. The original problem is to find y_p. Now we need to find u_1 and u_2, but we still have only one equation to solve, $L(y_p) = f$. The problem for u_1, u_2 cannot have a unique solution. So we are completely free to add a second equation to the original equation $L(y_p) = f$. We choose the second equation so that we can solve for u_1 and u_2. We unveil this second equation when we are in the middle of the proof of Theorem 2.4.4.

Proof of Theorem 2.4.4: We must find a function y_p solution of $L(y_p) = f$. We know a pair of fundamental solutions, y_1, y_2, of the homogeneous equation $L(y) = 0$. Here is where we generalize the reduction order method by looking for a function y_p of the form

$$y_p = u_1\, y_1 + u_2\, y_2,$$

where the functions u_1, u_2 must be determined from the equation $L(y_p) = f$. We started looking for one function, y_p, and now we are looking for two functions u_1, u_2. The original equation $L(y_p) = f$ will give us a relation between u_1 and u_2. Because we have added a new function to the problem, we need to add one more equation to the problem so we get a unique solution u_1, u_2. We are completely free to choose this extra equation. However, at this point we have no idea what to choose.

Before adding a new equation to the system, we work with the original equation. We introduce y_p into the equation $L(y_p) = f$. We must first compute the derivatives

$$y_p' = u_1'\,y_1 + u_1\,y_1' + u_2'\,y_2 + u_2\,y_2', \qquad y_p'' = u_1''\,y_1 + 2u_1'\,y_1' + u_1\,y_1'' + u_2''\,y_2 + 2u_2'\,y_2' + u_2\,y_2''.$$

We reorder a few terms and we see that $L(y_p) = f$ has the form

$$\begin{aligned} u_1''\,y_1 + u_2''\,y_2 + 2(u_1'\,y_1' + u_2\,y_2') + p\,(u_1'\,y_1 + u_2'\,y_2)& \\ +u_1\,(y_1'' + p\,y_1' + q\,y_1) + u_2\,(y_2'' + p\,y_2' + q\,y_2) &= f. \end{aligned}$$

The functions y_1 and y_2 are solutions to the homogeneous equation,

$$y_1'' + p\,y_1' + q\,y_1 = 0, \qquad y_2'' + p\,y_2' + q\,y_2 = 0,$$

so u_1 and u_2 must be solution of a simpler equation that the one above, given by

$$u_1''\,y_1 + u_2''\,y_2 + 2(u_1'\,y_1' + u_2'\,y_2') + p\,(u_1'\,y_1 + u_2'\,y_2) = f. \tag{2.4.4}$$

As we said above, this equation does not have a unique solution u_1, u_2. This is just a relation between these two functions. Here is where we need to add a new equation so that we can get a unique solution for u_1, u_2. What is an appropriate equation to add? Any equation that simplifies the Eq. (2.4.4) is a good candidate. For two reasons, we take the equation

$$u_1'\,y_1 + u_2'\,y_2 = 0. \tag{2.4.5}$$

The first reason is that this equation makes the last term on the righ-hand side of Eq. (2.4.4) vanish. So the system we need to solve is

$$u_1''\,y_1 + u_2''\,y_2 + 2(u_1'\,y_1' + u_2'\,y_2') = f \tag{2.4.6}$$

$$u_1'\,y_1 + u_2'\,y_2 = 0. \tag{2.4.7}$$

The second reason is that this second equation simplifies the first equation even further. Just take the derivative of the second equation,

$$\big(u_1'\,y_1 + u_2'\,y_2\big)' = 0 \quad \Rightarrow \quad u_1''\,y_1 + u_2''\,y_2 + (u_1'\,y_1' + u_2'\,y_2') = 0.$$

This last equation implies that the first three terms in Eq. (2.4.6) vanish identically, because of Eq.(2.4.7). So we end with the equations

$$\begin{aligned} u_1'\,y_1' + u_2'\,y_2' &= f \\ u_1'\,y_1 + u_2'\,y_2 &= 0. \end{aligned}$$

And this is a 2×2 algebraic linear system for the unknowns u_1', u_2'. It is hard to overstate the importance of the word "algebraic" in the previous sentence. From the second equation above we compute u_2' and we introduce it in the first equation,

$$u_2' = -\frac{y_1}{y_2}\,u_1' \quad \Rightarrow \quad u_1'y_1' - \frac{y_1y_2'}{y_2}\,u_1' = f \quad \Rightarrow \quad u_1'\Big(\frac{y_1'y_2 - y_1y_2'}{y_2}\Big) = f.$$

Recall that the Wronskian of two functions is $W_{y_1y_2} = y_1y_2' - y_1'y_2$, we get

$$u_1' = -\frac{y_2 f}{W_{y_1y_2}} \quad \Rightarrow \quad u_2' = \frac{y_1 f}{W_{y_1y_2}}.$$

These equations are the derivative of Eq. (2.4.3). Integrate them in the variable t and choose the integration constants to be zero. We get Eq. (2.4.3). This establishes the Theorem. □

Remark: The integration constants in the expressions for u_1, u_2 can always be chosen to be zero. To understand the effect of the integration constants in the function y_p, let us do the following. Denote by u_1 and u_2 the functions in Eq. (2.4.3), and given any real numbers c_1 and c_2 define

$$\tilde{u}_1 = u_1 + c_1, \qquad \tilde{u}_2 = u_2 + c_2.$$

Then the corresponding solution $\tilde{y}_p$ is given by

$$\tilde{y}_p = \tilde{u}_1\, y_1 + \tilde{u}_2\, y_2 = u_1\, y_1 + u_2\, y_2 + c_1\, y_1 + c_2\, y_2 \quad \Rightarrow \quad \tilde{y}_p = y_p + c_1\, y_1 + c_2\, y_2.$$

The two solutions $\tilde{y}_p$ and y_p differ by a solution to the homogeneous differential equation. So both functions are also solution to the nonhomogeneous equation. One is then free to choose the constants c_1 and c_2 in any way. We chose them in the proof above to be zero.

Example 2.4.6: Find the general solution of the nonhomogeneous equation

$$y'' - 5y' + 6y = 2e^t.$$

Solution: The formula for y_p in Theorem 2.4.4 requires we know fundamental solutions to the homogeneous problem. So we start finding these solutions first. Since the equation has constant coefficients, we compute the characteristic equation,

$$r^2 - 5r + 6 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big(5 \pm \sqrt{25-24}\big) \quad \Rightarrow \quad \begin{cases} r_+ = 3, \\ r_- = 2. \end{cases}$$

So, the functions y_1 and y_2 in Theorem 2.4.4 are in our case given by

$$y_1(t) = e^{3t}, \qquad y_2(t) = e^{2t}.$$

The Wronskian of these two functions is given by

$$W_{y_1y_2}(t) = (e^{3t})(2e^{2t}) - (3e^{3t})(e^{2t}) \quad \Rightarrow \quad W_{y_1y_2}(t) = -e^{5t}.$$

We are now ready to compute the functions u_1 and u_2. Notice that Eq. (2.4.3) the following differential equations

$$u_1' = -\frac{y_2 f}{W_{y_1y_2}}, \qquad u_2' = \frac{y_1 f}{W_{y_1y_2}}.$$

So, the equation for u_1 is the following,

$$u_1' = -e^{2t}(2e^t)(-e^{-5t}) \quad \Rightarrow \quad u_1' = 2e^{-2t} \quad \Rightarrow \quad u_1 = -e^{-2t},$$

$$u_2' = e^{3t}(2e^t)(-e^{-5t}) \quad \Rightarrow \quad u_2' = -2e^{-t} \quad \Rightarrow \quad u_2 = 2e^{-t},$$

where we have chosen the constant of integration to be zero. The particular solution we are looking for is given by

$$y_p = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t}) \quad \Rightarrow \quad y_p = e^t.$$

Then, the general solution theorem for nonhomogeneous equation implies

$$y_{\text{gen}}(t) = c_+\, e^{3t} + c_-\, e^{2t} + e^t \qquad c_+, c_- \in \mathbb{R}.$$

$\lhd$

Example 2.4.7: Find a particular solution to the differential equation

$$t^2 y'' - 2y = 3t^2 - 1,$$

knowing that $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2y'' - 2y = 0$.

SOLUTION: We first rewrite the nonhomogeneous equation above in the form given in Theorem 2.4.4. In this case we must divide the whole equation by t^2,

$$y'' - \frac{2}{t^2}\, y = 3 - \frac{1}{t^2} \quad \Rightarrow \quad f(t) = 3 - \frac{1}{t^2}.$$

We now proceed to compute the Wronskian of the fundamental solutions y_1, y_2,

$$W_{y_1 y_2}(t) = (t^2)\Big(\frac{-1}{t^2}\Big) - (2t)\Big(\frac{1}{t}\Big) \quad \Rightarrow \quad W_{y_1 y_2}(t) = -3.$$

We now use the equation in (2.4.3) to obtain the functions u_1 and u_2,

$$u_1' = -\frac{1}{t}\left(3 - \frac{1}{t^2}\right)\frac{1}{-3} \qquad\qquad u_2' = (t^2)\left(3 - \frac{1}{t^2}\right)\frac{1}{-3}$$
$$= \frac{1}{t} - \frac{1}{3}\,t^{-3} \quad \Rightarrow \quad u_1 = \ln(t) + \frac{1}{6}t^{-2}, \qquad = -t^2 + \frac{1}{3} \quad \Rightarrow \quad u_2 = -\frac{1}{3}\,t^3 + \frac{1}{3}t.$$

A particular solution to the nonhomogeneous equation above is $\tilde{y}_p = u_1 y_1 + u_2 y_2$, that is,

$$\begin{aligned}
\tilde{y}_p &= \Big[\ln(t) + \frac{1}{6}t^{-2}\Big](t^2) + \frac{1}{3}(-t^3 + t)(t^{-1}) \\
&= t^2 \ln(t) + \frac{1}{6} - \frac{1}{3}\,t^2 + \frac{1}{3} \\
&= t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\,t^2 \\
&= t^2 \ln(t) + \frac{1}{2} - \frac{1}{3}\,y_1(t).
\end{aligned}$$

However, a simpler expression for a solution of the nonhomogeneous equation above is

$$y_p = t^2 \ln(t) + \frac{1}{2}.$$

◁

Remark: Sometimes it could be difficult to remember the formulas for functions u_1 and u_2 in (2.4.3). In such case one can always go back to the place in the proof of Theorem 2.4.4 where these formulas come from, the system

$$\begin{aligned}
u_1' y_1' + u_2' y_2' &= f \\
u_1' y_1 + u_2' y_2 &= 0.
\end{aligned}$$

The system above could be simpler to remember than the equations in (2.4.3). We end this Section using the equations above to solve the problem in Example 2.4.7. Recall that the solutions to the homogeneous equation in Example 2.4.7 are $y_1(t) = t^2$, and $y_2(t) = 1/t$, while the source function is $f(t) = 3 - 1/t^2$. Then, we need to solve the system

$$\begin{aligned}
t^2\, u_1' + u_2' \frac{1}{t} &= 0, \\
2t\, u_1' + u_2' \frac{(-1)}{t^2} &= 3 - \frac{1}{t^2}.
\end{aligned}$$

This is an algebraic linear system for u_1' and u_2'. Those are simple to solve. From the equation on top we get u_2' in terms of u_1', and we use that expression on the bottom equation,

$$u_2' = -t^3\, u_1' \quad \Rightarrow \quad 2t\, u_1' + t\, u_1' = 3 - \frac{1}{t^2} \quad \Rightarrow \quad u_1' = \frac{1}{t} - \frac{1}{3t^3}.$$

Substitue back the expression for u_1' in the first equation above and we get u_2'. We get,

$$u_1' = \frac{1}{t} - \frac{1}{3t^3}$$
$$u_2' = -t^2 + \frac{1}{3}.$$

We should now integrate these functions to get u_1 and u_2 and then get the particular solution $\tilde{y}_p = u_1 y_1 + u_2 y_2$. We do not repeat these calculations, since they are done Example 2.4.7.

2.4.4. **Exercises.**

2.4.1.- .

2.4.2.- .

2.5. Applications

Different physical systems are mathematically identical. In this Section we show that a weight attached to a spring, oscillating trough air of under water, is mathematically identical to the behavior of an electric current in a circuit containing a resistor, a capacitor, and an inductor. Mathematical identical means in this case that both systems are described by the same differential equation.

2.5.1. **Review of Constant Coefficient Equations.**

In § 2.3 we have found solutions to second order, linear, homogeneous, differential equations with constant coefficients,

$$y'' + a_1\, y' + a_0\, y = 0, \qquad a_1, a_0 \in \mathbb{R}. \tag{2.5.1}$$

Theorem 2.3.2 contains formulas for the general solution of this equation. We review these formulas here and at the same time we introduce new names that are common in the physics literature to describe these solutions. The first step to obtain solutions to Eq. (2.5.1) is to find the roots or the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$, which are given by

$$r_{\pm} = -\frac{a_1}{2} \pm \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

We then have three different cases to consider.

(a) A system is called *overdamped* in the case that $a_1^2 - 4a_0 > 0$. In this case the characteristic polynomial has real and distinct roots, r_+, r_-, and the corresponding solutions to the differential equation are

$$y_+(t) = e^{r_+ t}, \qquad y_-(t) = e^{r_- t}.$$

So the solutions are exponentials, increasing or decreasing, according whether the roots are positive or negative, respectively. The decreasing exponential solutions originate the name overdamped solutions.

(b) A system is called *critically damped* in the case that $a_1^2 - 4a_0 = 0$. In this case the characteristic polynomial has only one real, repeated, root, $r_0 = -a_1/2$, and the corresponding solutions to the differential equation are then,

$$y_+(t) = e^{-a_1 t/2}, \qquad y_-(t) = t\, e^{-a_1 t/2}.$$

(c) A system is called *underdamped* in the case that $a_1^2 - 4a_0 < 0$. In this case the characteristic polynomial has two complex roots, $r_{\pm} = \alpha \pm \beta i$, one being the complex conjugate of the other, since the polynomial has real coefficients. The corresponding solutions to the differential equation are

$$y_+(t) = e^{\alpha t}\cos(\beta t), \qquad y_-(t) = e^{\alpha t}\sin(\beta t).$$

where $\alpha = -\dfrac{a_1}{2}$ and $\beta = \dfrac{1}{2}\sqrt{4a_0 - a_1^2}$. In the particular case that the real part of the solutions vanishes, $a_1 = 0$, the system is called *undamped*, since it has oscillatory solutions without any exponential decay or increase.

2.5.2. **Undamped Mechanical Oscillations.**

Springs are curious objects, when you slightly deform them they create a force proportional and in opposite direction to the deformation. When you release the spring, it goes back to its original shape. This is true for small enough deformations. If you stretch the spring long enough, the deformations are permanent.

Consider a spring-plus-body system as shown in Fig. 2.5.2. A spring is fixed to a ceiling and hangs vertically with a natural length l. It stretches by Δl when a body with mass m is attached to the spring lower end, just like the middle spring in Fig. 2.5.2. We assume that the weight m is small enough so that the spring is not damaged. This means that the spring acts like a normal spring, whenever it is deformed by an amount Δl it makes a force proportional and opposite to the deformation, $F_{s0} = -k\,\Delta l$. Here $k > 0$ is a constant that depends on the type of spring. Newton's law of motion implies the following result.

Theorem 2.5.1. *A spring-plus-body system with spring constant k, body mass m, at rest with a spring deformation Δl, within the range where the spring acts like a spring, satisfies*

$$mg = k\,\Delta l.$$

Proof of Theorem 2.5.1: Since the spring-plus-body system is at rest, Newton's law of motion implies that all forces acting on the body must add up to zero. The only two forces acting on the body are its weight, $F_g = mg$, and the force done by the spring, $F_{s0} = -k\,\Delta l$. We have used the hypothesis that Δl is small enough so the spring is not damaged. We are using the sign convention displayed in Fig. 2.5.2, where forces downwards are positive. As we said above, since the body is at rest, the addition of all forces acting on the body must vanish,

$$F_g + F_{s0} = 0 \quad \Rightarrow \quad mg = k\,\Delta l.$$

This establishes the Theorem. □

Rewriting the equation above as

$$k = \frac{mg}{\Delta l}.$$

it is possible to compute the spring constant k by measuring the displacement Δl and knowing the body mass m.

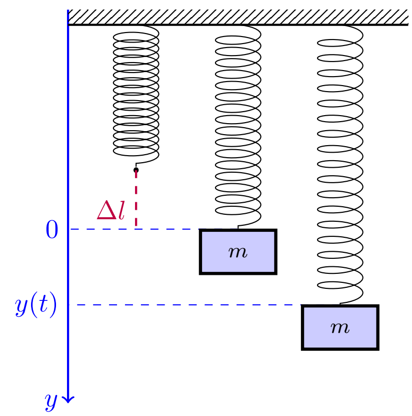


FIGURE 11. Springs with weights.

We now find out how the body will move when we take it away from the rest position. To describe that movement we introduce a vertical coordinate for the displacements, y, as shown in Fig. 2.5.2, with y positive downwards, and $y = 0$ at the rest position of the spring and the body. The physical system we want to describe is simple; we further stretch the spring with the body by y_0 and then we release it with an initial velocity $\hat{y}_0$. Newton's law of motion determine the subsequent motion.

Theorem 2.5.2. *The vertical movement of a spring-plus-body system in air with spring constant $k > 0$ and body mass $m > 0$ is described by the solutions of the differential equation*

$$m\,y''(t) + k\,y(t) = 0, \tag{2.5.2}$$

where y is the vertical displacement function as shown in Fig. 2.5.2. Furthermore, there is a unique solution to Eq. (2.5.2) satisfying the initial conditions $y(0) = y_0$ and $y'(0) = y_1$,

$$y(t) = A\,\cos(\omega_0 t - \phi),$$

with natural frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$, where the amplitude $A \geqslant 0$ and phase-shift $\phi \in (-\pi, \pi]$,

$$A = \sqrt{y_0^2 + \frac{y_1^2}{\omega_0^2}}, \qquad \phi = \arctan\Big(\frac{y_1}{\omega_0 y_0}\Big).$$

Remark: The natural frequency of the system $\omega_0 = \sqrt{k/m}$ is an angular, or circular, frequency. So when $\omega_0 \neq 0$ the motion of the system is periodic with *period* $T = 2\pi/\omega_0$ and *frequency* $\nu_0 = \omega_0/(2\pi)$.

Proof of Theorem 2.5.2: Newton's second law of motion says that mass times acceleration of the body $m\,y''(t)$ must be equal to the sum of all forces acting on the body, hence

$$m\,y''(t) = F_g + F_{s0} + F_s(t),$$

where $F_s(t) = -k\,y(t)$ is the force done by the spring due to the extra displacement y. Since the first two terms on the right hand side above cancel out, $F_g + F_{s0} = 0$, the body displacement from the equilibrium position, $y(t)$, must be solution of the differential equation

$$m\,y''(t) + k\,y(t) = 0.$$

which is Eq. (2.5.2). In § 2.3 we have seen how to solve this type of differential equations. The characteristic polynomial is $p(r) = mr^2 + k$, which has complex roots $r_\pm = \pm\omega_0^2\,i$, where we introduced the natural frequency of the system,

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

The reason for this name is the calculations done in § 2.3, where we found that a real-valued expression for the general solution to Eq. (2.5.2) is given by

$$y_{\rm gen}(t) = c_+\,\cos(\omega_0 t) + c_-\,\sin(\omega_0 t).$$

This means that the body attached to the spring oscillates around the equilibrium position $y = 0$ with period $T = 2\pi/\omega_0$, hence frequency $\nu_0 = \omega_0/(2\pi)$. There is an equivalent way to express the general solution above given by

$$y_{\rm gen}(t) = A\,\cos(\omega_0 t - \phi).$$

These two expressions for $y_{\rm gen}$ are equivalent because of the trigonometric identity

$$A\cos(\omega_0 t - \phi) = A\cos(\omega_0 t)\cos(\phi) + A\sin(\omega_0 t)\sin(\phi),$$

which holds for all A and ϕ, and $\omega_0 t$. Then, it is not difficult to see that

$$\left.\begin{aligned} c_+ &= A\cos(\phi),\\ c_- &= A\sin(\phi). \end{aligned}\right\} \quad\Leftrightarrow\quad \left\{\begin{aligned} A &= \sqrt{c_+^2 + c_-^2},\\ \phi &= \arctan\Big(\frac{c_-}{c_+}\Big). \end{aligned}\right.$$

Since both expressions for the general solution are equivalent, we use the second one, in terms of the amplitude and phase-shift. The initial conditions $y(0) = y_0$ and $y'(0) = y_1$ determine the constants A and ϕ. Indeed,

$$\left.\begin{aligned} y_0 &= y(0) = A\,\cos(\phi),\\ y_1 &= y'(0) = A\omega_0\,\sin(\phi). \end{aligned}\right\} \quad\Rightarrow\quad \left\{\begin{aligned} A &= \sqrt{y_0^2 + \frac{y_1^2}{\omega_0^2}},\\ \phi &= \arctan\Big(\frac{y_1}{\omega_0 y_0}\Big). \end{aligned}\right.$$

This establishes the Theorem. □

EXAMPLE 2.5.1: Find the movement of a 50 gr mass attached to a spring moving in air with initial conditions $y(0) = 4\,\text{cm}$ and $y'(0) = 40\,\text{cm/s}$. The spring is such that a 30 gr mass stretches it 6 cm. Approximate the acceleration of gravity by $1000\,\text{cm/s}^2$.

SOLUTION: Theorem 2.5.2 says that the equation satisfied by the displacement y is given by

$$my'' + ky = 0.$$

In order to solve this equation we need to find the spring constant, k, which by Theorem 2.5.1 is given by $k = mg/\Delta l$. In our case when a mass of $m = 30\,\text{gr}$ is attached to the sprint, it stretches $\Delta l = 6\,\text{cm}$, so we get,

$$k = \frac{(30)\,(1000)}{6} \quad \Rightarrow \quad k = 5000\,\frac{\text{gr}}{\text{s}^2}.$$

Knowing the spring constant k we can now describe the movement of the body with mass $m = 50\,\text{gr}$. The solution of the differential equation above is obtained as usual, first find the roots of the characteristic polynomial

$$mr^2 + k = 0 \quad \Rightarrow \quad r_\pm = \pm\omega_0 i, \quad \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000}{50}} \quad \Rightarrow \quad \omega_0 = 10\,\frac{1}{\text{s}}.$$

We write down the general solution in terms of the amplitude A and phase-shift ϕ,

$$y(t) = A\,\cos(\omega_0 t - \phi) \quad \Rightarrow \quad y(t) = A\,\cos(10\,t - \phi).$$

To accommodate the initial conditions we need the function $y'(t) = -A\omega_0\,\sin(\omega_0 t - \phi)$. The initial conditions determine the amplitude and phase-shift, as follows,

$$\left.\begin{aligned} 4 = y(0) &= A\cos(\phi), \\ 40 = y'(0) &= -10\,A\sin(-\phi) \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} &A = \sqrt{16+16}, \\ &\phi = \arctan\Big(\frac{40}{(10)(4)}\Big). \end{aligned}\right.$$

We obtain that $A = 4\sqrt{2}$ and $\tan(\phi) = 1$. The later equation implies that either $\phi = \pi/4$ or $\phi = -3\pi/4$, for $\phi \in (-\pi, \pi]$. If we pick the second value, $\phi = -3\pi/4$, this would imply that $y(0) < 0$ and $y'(0) < 0$, which is not true in our case. So we **must pick** the value $\phi = \pi/4$. We then conclude:

$$y(t) = 4\sqrt{2}\,\cos\Big(10\,t - \frac{\pi}{4}\Big).$$

◁

2.5.3. **Damped Mechanical Oscillations.** Suppose now that the body in the spring-plus-body system is a thin square sheet of metal. If the main surface of the sheet is perpendicular to the direction of motion, then the air dragged by the sheet during the spring oscillations will be significant enough to slow down the spring oscillations in an appreciable time. One can find out that the friction force done by the air opposes the movement and it is proportional to the velocity of the body, that is, $F_d = -d\,y'(t)$. We call such force a *damping force*, where $d > 0$ is the *damping constant*, and systems having such force damped systems. We now describe the spring-plus-body system in the case that there is a non-zero damping force.

Theorem 2.5.3.

(a) *The vertical displacement y, function as shown in Fig. 2.5.2, of a spring-plus-body system with spring constant $k > 0$, body mass $m > 0$, and damping constant $d \geqslant 0$, is described by the solutions of*

$$m\,y''(t) + d\,y'(t) + k\,y(t) = 0, \tag{2.5.3}$$

(b) *The roots of the characteristic polynomial of Eq. (2.5.3) are $r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}$, with damping frequency $\omega_d = \dfrac{d}{2m}$ and natural frequency $\omega_0 = \sqrt{\dfrac{k}{m}}$.*

(c) *The solutions to Eq. (2.5.3) fall into one of the following cases:*

(i) *A system with $\omega_d > \omega_0$ is called overdamped, with general solution to Eq. (2.5.3)*

$$y_{\text{gen}}(t) = c_+ \, e^{r_+ t} + c_- \, e^{r_- t}.$$

(ii) *A system with $\omega_d = \omega_0$ is called critically damped, with general solution to Eq. (2.5.3)*

$$y_{\text{gen}}(t) = c_+ \, e^{-\omega_d t} + c_- \, t \, e^{-\omega_d t}.$$

(iii) *A system with $\omega_d < \omega_0$ is called underdamped, with general solution to Eq. (2.5.3)*

$$y_{\text{gen}}(t) = A \, e^{-\omega_d t} \cos(\beta t - \phi),$$

where $\beta = \sqrt{\omega_0^2 - \omega_d^2}$ is the system frequency. The case $\omega_d = 0$ is called undamped.

(d) *There is a unique solution to Eq. (2.5.2) with initial conditions $y(0) = y_0$ and $y'(0) = y_1$.*

Remark: In the case the damping constant vanishes we recover Theorem 2.5.2.

Proof of Therorem 2.5.2: Newton's second law of motion says that mass times acceleration of the body $m \, y''(t)$ must be equal to the sum of all forces acting on the body. In the case that we take into account the air dragging force we have

$$m \, y''(t) = F_g + F_{s0} + F_s(t) + F_d(t),$$

where $F_s(t) = -k \, y(t)$ as in Theorem 2.5.2, and $F_d(t) = -d \, y'(t)$ is the air-body dragging force. Since the first two terms on the right hand side above cancel out, $F_g + F_{s0} = 0$, as mentioned in Theorem 2.5.1, the body displacement from the equilibrium position, $y(t)$, must be solution of the differential equation

$$m \, y''(t) + +d \, y'(t) + k \, y(t) = 0.$$

which is Eq. (2.5.3). In § 2.3 we have seen how to solve this type of differential equations. The characteristic polynomial is $p(r) = mr^2 + dr + k$, which has complex roots

$$r_\pm = \frac{1}{2m}\big[-d \pm \sqrt{d^2 - 4mk}\big] = -\frac{d}{2m} \pm \sqrt{\Big(\frac{d}{2m}\Big)^2 - \frac{k}{m}} \quad \Rightarrow \quad r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}.$$

where $\omega_d = \dfrac{d}{2m}$ and $\omega_0 = \sqrt{\dfrac{k}{m}}$. In § 2.3 we found that the general solution of a differential equation with a characteristic polynomial having roots as above can be divided into three groups. For the case $r_+ \neq r_-$ real valued, we obtain case (ci), for the case $r_+ = r_-$ we obtain case (cii). Finally, we said that the general solution for the case of two complex roots $r_\pm = \alpha + \beta i$ was given by

$$y_{\text{gen}}(t) = e^{\alpha t}\big(c_+ \cos(\beta t) + c_- \sin(\beta t)\big).$$

In our case $\alpha = -\omega_d$ and $\beta = \sqrt{\omega_0^2 - \omega_d^2}$. We now rewrite the second factor on the right-hand side above in terms of an amplitude and a phase shift,

$$y_{\text{gen}}(t) = A \, e^{-\omega_d t} \cos(\beta t - \phi).$$

The main result from § 2.3 says that the initial value problem in Theorem 2.5.3 has a unique solution for each of the three cases above. This establishes the Theorem. □

Example 2.5.2: Find the movement of a 5Kg mass attached to a spring with constant $k = 5\text{Kg/Secs}^2$ moving in a medium with damping constant $d = 5\text{Kg/Secs}$, with initial conditions $y(0) = \sqrt{3}$ and $y'(0) = 0$.

Solution: By Theorem 2.5.3 the differential equation for this system is $my'' + dy' + ky = 0$, with $m = 5$, $k = 5$, $d = 5$. The roots of the characteristic polynomial are

$$r_\pm = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}, \quad \omega_d = \frac{d}{2m} = \frac{1}{2}, \quad \omega_0 = \sqrt{\frac{k}{m}} = 1,$$

that is,

$$r_{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

This means our system has underdamped oscillations. Following Theorem 2.5.3 part (ciii), our solution must be given by

$$y(t) = A\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big).$$

We only need to introduce the initial conditions into the expression for y to find out the amplitude A and phase-shift ϕ. In order to do that we first compute the derivative,

$$y'(t) = -\frac{1}{2}\, A\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big) - \frac{\sqrt{3}}{2}\, A\, e^{-t/2} \sin\Big(\frac{\sqrt{3}}{2}\, t - \phi\Big).$$

The initial conditions in the example imply,

$$\sqrt{3} = y(0) = A\cos(\phi), \qquad 0 = y'(0) = -\frac{1}{2}\, A\cos(\phi) + \frac{\sqrt{3}}{2}\, A\sin(\phi).$$

The second equation above allows us to compute the phase-shift, since

$$\tan(\phi) = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \phi = \frac{\pi}{6}, \quad \text{or} \quad \phi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}.$$

If $\phi = -5\pi/6$, then $y(0) < 0$, which is not out case. Hence we **must choose** $\phi = \pi/6$. With that phase-shift, the amplitude is given by

$$\sqrt{3} = A\cos\Big(\frac{\pi}{6}\Big) = A\,\frac{\sqrt{3}}{2} \quad \Rightarrow \quad A = 2.$$

We conclude: $y(t) = 2\, e^{-t/2} \cos\Big(\frac{\sqrt{3}}{2}\, t - \frac{\pi}{6}\Big)$. ◁

2.5.4. **Electrical Oscillations.** We describe the electric current flowing through an electric circuit consisting of a resistor, a capacitor, and an inductor connected in series as shown in Fig. 12. A current can start when a magnet is moved near the inductor. If the circuit has low resistance, the current will keep flowing through the inductor between the capacitor plates, endlessly. There is no need of a power source to keep the current flowing. The presence of a resistance transforms the current energy into heat, damping the current oscillation.

This system, called RLC circuit, is described by an integro-differential equation found by Kirchhoff, now called Kirchhoff's voltage law, relating the *resistor* R, *capacitor* C, *inductor* L, and the current I in a circuit as follows,

$$L\, I'(t) + R\, I(t) + \frac{1}{C} \int_{t_0}^{t} I(s)\, ds = 0. \quad (2.5.4)$$

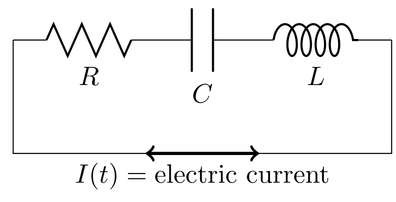


FIGURE 12. An RLC circuit.

Kirchhoff's voltage law is all we need to present the following result.

Theorem 2.5.4. *The electric current function I in an RLC circuit with resistance $R \geqslant 0$, capacitance $C > 0$, and inductance $L > 0$, satisfies the differential equation*

$$L\, I'' + R\, I' + \frac{1}{C}\, I = 0.$$

Furthermore, the results in Theorem 2.5.3 parts (c), (d), hold with the roots of the characteristic polynomial $r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}$, *and with the damping frequency* $\omega_d = \dfrac{R}{2L}$ *and natural frequency* $\omega_0 = \sqrt{\dfrac{1}{LC}}$.

Proof of Theorem 2.5.4: Compute the derivate on both sides in Eq. (2.5.4),

$$L\,I'' + R\,I' + \frac{1}{C}\,I = 0,$$

and divide by L,

$$I''(t) + 2\Big(\frac{R}{2L}\Big)\,I'(t) + \frac{1}{LC}\,I(t) = 0.$$

Introduce $\omega_d = \dfrac{R}{2L}$ and $\omega_0 = \dfrac{1}{\sqrt{LC}}$, then Kirchhoff's law can be expressed as the second order, homogeneous, constant coefficients, differential equation

$$I'' + 2\omega_d\,I' + \omega_0^2\,I = 0.$$

The rest of the proof follows the one of Theorem 2.5.3. This establishes the Theorem. □

Example 2.5.3: Find real-valued fundamental solutions to $I'' + 2\omega_d\,I' + \omega_0^2\,I = 0$, where $\omega_d = R/(2L)$, $\omega_0^2 = 1/(LC)$, in the cases (a), (b) below.

Solution: The roots of the characteristic polynomial, $p(r) = r^2 + 2\omega_d r + \omega_0^2$, are given by

$$r_{\pm} = \frac{1}{2}\big[-2\omega_d \pm \sqrt{4\omega_d^2 - 4\omega_0^2}\big] \quad \Rightarrow \quad r_{\pm} = -\omega_d \pm \sqrt{\omega_d^2 - \omega_0^2}.$$

Case (a): $R = 0$. This implies $\omega_d = 0$, so $r_{\pm} = \pm i\omega_0$. Therefore,

$$I_1(t) = \cos(\omega_0 t), \qquad I_2(t) = \sin(\omega_0 t).$$

Remark: When the circuit has no resistance, the current oscillates without dissipation.

Case (b): $R < \sqrt{4L/C}$. This implies

$$R^2 < \frac{4L}{C} \;\Leftrightarrow\; \frac{R^2}{4L^2} < \frac{1}{LC} \;\Leftrightarrow\; \omega_d^2 < \omega_0^2.$$

Therefore, the characteristic polynomial has complex roots $r_{\pm} = -\omega_d \pm i\sqrt{\omega_0^2 - \omega_d^2}$, hence the fundamental solutions are

$$I_1(t) = e^{-\omega_d t}\,\cos(\beta\,t),$$

$$I_2(t) = e^{-\omega_d t}\,\sin(\beta\,t),$$

with $\beta = \sqrt{\omega_0^2 - \omega_d^2}$. Therefore, the resistance R damps the current oscillations produced by the capacitor and the inductance. ◁

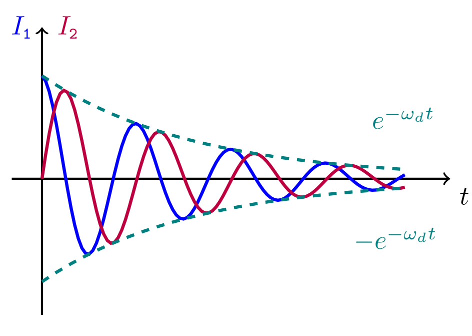


Figure 13. Typical currents I_1, I_2 for case (b).

2.5.5. **Exercises.**

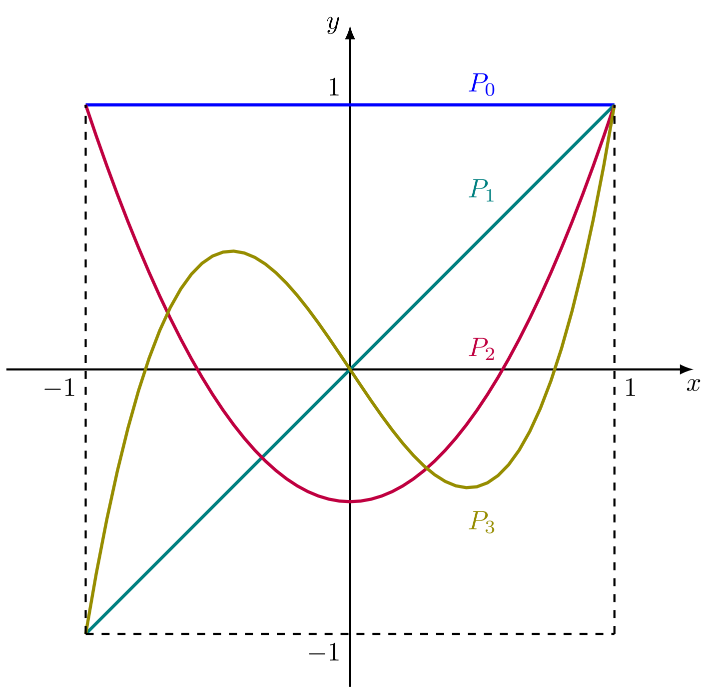
2.5.1.- .

2.5.2.- .

Chapter 3. Power Series Solutions

The first differential equations were solved around the end of the seventeen century and beginning of the eighteen century. We studied a few of these equations in § 1.1-1.4 and the constant coefficients equations in Chapter 2. By the middle of the eighteen century people realized that the methods we learnt in these first sections had reached a dead end. One reason was the lack of functions to write the solutions of differential equations. The elementary functions we use in calculus, such as polynomials, quotient of polynomials, trigonometric functions, exponentials, and logarithms, were simply not enough. People even started to think of differential equations as sources to find new functions. It was matter of little time before mathematicians started to use power series expansions to find solutions of differential equations. Convergent power series define functions far more general than the elementary functions from calculus.

In § 3.1 we study the simplest case, when the power series is centered at a regular point of the equation. The coefficients of the equation are analytic functions at regular points, in particular continuous. In § 3.2 we study the Euler equidimensional equation. The coefficients of an Euler equation diverge at a particular point in a very specific way. No power series are needed to find solutions in this case. In § 3.3 we solve equations with regular singular points. The equation coefficients diverge at regular singular points in a way similar to the coefficients in an Euler equation. We will find solutions to these equations using the solutions to an Euler equation and power series centered precisely at the regular singular points of the equation.



3.1. Solutions Near Regular Points

We study second order linear homogeneous differential equations with variable coefficients,

$$y'' + p(x)\, y' + q(x)\, y = 0.$$

We look for solutions on a domain where the equation coefficients p, q are analytic functions. Recall that a function is analytic on a given domain iff it can be written as a convergent power series expansions on that domain. In Appendix B we review a few ideas on analytic functions and power series expansion that we need in this section. A regular point of the equation is every point where the equation coefficients are analytic. We look for solutions that can be written as power series centered at a regular point. For simplicity we solve only homogeneous equations, but the power series method can be used with nonhomogeneous equations without introducing substantial modifications.

3.1.1. Regular Points.

We now look for solutions to second order linear homogeneous differential equations having variable coefficients. Recall we solved the constant coefficient case in Chapter 2. We have seen that the solutions to constant coefficient equations can be written in terms of elementary functions such as quotient of polynomials, trigonometric functions, exponentials, and logarithms. For example, the equation

$$y'' + y = 0$$

has the fundamental solutions $y_1(x) = \cos(x)$ and $y_2(x) = \sin(x)$. But the equation

$$x\, y'' + y' + x\, y = 0$$

cannot be solved in terms of elementary functions, that is in terms of quotients of polynomials, trigonometric functions, exponentials and logarithms. Except for equations with constant coefficient and equations with variable coefficient that can be transformed into constant coefficient by a change of variable, no other second order linear equation can be solved in terms of elementary functions. Still, we are interested in finding solutions to variable coefficient equations. Mainly because these equations appear in the description of so many physical systems.

We have said that power series define more general functions than the elementary functions mentioned above. So we look for solutions using power series. In this section we center the power series at a regular point of the equation.

Definition 3.1.1. *A point $x_0 \in \mathbb{R}$ is called a **regular point** of the equation*

$$y'' + p(x)\, y' + q(x)\, y = 0, \tag{3.1.1}$$

*iff p, q are analytic functions at x_0. Otherwise x_0 is called a **singular point** of the equation.*

Remark: Near a regular point x_0 the coefficients p and q in the differential equation above can be written in terms of power series centered at x_0,

$$p(x) = p_0 + p_1\,(x - x_0) + p_2\,(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} p_n\,(x - x_0)^n,$$

$$q(x) = q_0 + q_1\,(x - x_0) + q_2\,(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} q_n\,(x - x_0)^n,$$

and these power series converge in a neighborhood of x_0.

Example 3.1.1: Find all the regular points of the equation

$$x\, y'' + y' + x^2\, y = 0.$$

SOLUTION: We write the equation in the form of Eq. (3.1.1),

$$y'' + \frac{1}{x}\,y' + x\,y = 0.$$

In this case the coefficient functions are $p(x) = 1/x$, and $q(x) = x$. The function q is analytic in $\mathbb{R}$. The function p is analytic for all points in $\mathbb{R} - \{0\}$. So the point $x_0 = 0$ is a singular point of the equation. Every other point is a regular point of the equation. ◁

3.1.2. **The Power Series Method.** The differential equation in (3.1.1) is a particular case of the equations studied in § 2.1, and the existence result in Theorem 2.1.2 applies to Eq. (3.1.1). This Theorem was known to Lazarus Fuchs, who in 1866 added the following: If the coefficient functions p and q are analytic on a domain, so is the solution on that domain. Fuchs went ahead and studied the case where the coefficients p and q have singular points, which we study in § 3.3. The result for analytic coefficients is summarized below.

Theorem 3.1.2. *If the functions p, q are analytic on an open interval $(x_0 - \rho, x_0 + \rho) \subset \mathbb{R}$, then the differential equation*

$$y'' + p(x)\,y' + q(x)\,y = 0,$$

has two independent solutions, y_1, y_2, which are analytic on the same interval.

Remark: A complete proof of this theorem can be found in [2], Page 169. See also [10], § 29. We present the first steps of the proof and we leave the convergence issues to the latter references. The proof we present is based on power series expansions for the coefficients p, q, and the solution y. This is not the proof given by Fuchs in 1866.

Proof of Thorem 3.1.2: Since the coefficient functions p and q are analytic functions on $(x_0 - \rho, x_0 + \rho)$, where $\rho > 0$, they can be written as power series centered at x_0,

$$p(x) = \sum_{n=0}^{\infty} p_n\,(x - x_0)^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n\,(x - x_0)^n.$$

We look for solutions that can also be written as power series expansions centered at x_0,

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^n.$$

We start computing the first derivatives of the function y,

$$y'(x) = \sum_{n=0}^{\infty} n a_n\,(x - x_0)^{(n-1)} \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n a_n\,(x - x_0)^{(n-1)},$$

where in the second expression we started the sum at $n = 1$, since the term with $n = 0$ vanishes. Relabel the sum with $m = n - 1$, so when $n = 1$ we have that $m = 0$, and $n = m + 1$. Therefore, we get

$$y'(x) = \sum_{m=0}^{\infty} (m + 1) a_{(m+1)}\,(x - x_0)^m.$$

We finally rename the summation index back to n,

$$y'(x) = \sum_{n=0}^{\infty} (n + 1) a_{(n+1)}\,(x - x_0)^n. \tag{3.1.2}$$

From now on we do these steps at once, and the notation $n - 1 = m \to n$ means

$$y'(x) = \sum_{n=1}^{\infty} n a_n\,(x - x_0)^{(n-1)} = \sum_{n=0}^{\infty} (n + 1) a_{(n+1)}\,(x - x_0)^n.$$

We continue computing the second derivative of function y,

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n\,(x-x_0)^{(n-2)},$$

and the transformation $n-2 = m \to n$ gives us the expression

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}\,(x-x_0)^n.$$

The idea now is to put all these power series back in the differential equation. We start with the term

$$\begin{aligned} q(x)\,y &= \Big(\sum_{n=0}^{\infty} q_n(x-x_0)^n\Big)\,\Big(\sum_{m=0}^{\infty} a_m(x-x_0)^m\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k q_{n-k}\Big)\,(x-x_0)^n, \end{aligned}$$

where the second expression above comes from standard results in power series multiplication. A similar calculation gives

$$\begin{aligned} p(x)\,y' &= \Big(\sum_{n=0}^{\infty} p_n(x-x_0)^n\Big)\,\Big(\sum_{m=0}^{\infty} (m+1)a_{(m+1)}(x-x_0)^m\Big) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} (k+1)a_{(k+1)}p_{n-k}\Big)\,(x-x_0)^n. \end{aligned}$$

Therefore, the differential equation $y'' + p(x)\,y' + q(x)\,y = 0$ has now the form

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)a_{(n+2)} + \sum_{k=0}^{n}\big[(k+1)a_{(k+1)}p_{(n-k)} + a_k q_{(n-k)}\big]\Big]\,(x-x_0)^n = 0.$$

So we obtain a *recurrence relation* for the coefficients a_n,

$$(n+2)(n+1)a_{(n+2)} + \sum_{k=0}^{n}\big[(k+1)a_{(k+1)}p_{(n-k)} + a_k q_{(n-k)}\big] = 0,$$

for $n = 0, 1, 2, \cdots$. Equivalently,

$$a_{(n+2)} = -\frac{1}{(n+2)(n+1)}\,\sum_{k=0}^{n}\big[(k+1)a_{(k+1)}p_{(n-k)} + a_k q_{(n-k)}. \tag{3.1.3}$$

We have obtained an expression for $a_{(n+2)}$ in terms of the previous coefficients $a_{(n+1)}, \cdots, a_0$ and the coefficients of the function p and q. If we choose arbitrary values for the first two coefficients a_0 and a_1, the the recurrence relation in (3.1.3) define the remaining coefficients $a_2, a_3, \cdots$ in terms of a_0 and a_1. The coefficients a_n chosen in such a way guarantee that the function y defined in (3.1.2) satisfies the differential equation.

In order to finish the proof of Theorem 3.1.2 we need to show that the power series for y defined by the recurrence relation actually converges on a nonempty domain, and furthermore that this domain is the same where p and q are analytic. This part of the proof is too complicated for us. The interested reader can find the rest of the proof in [2], Page 169. See also [10], § 29. □

It is important to understand the main ideas in the proof above, because we will follow these ideas to find power series solutions to differential equations. So we now summarize the main steps in the proof above:

(a) Write a power series expansion of the solution centered at a regular point x_0,

$$y(x) = \sum_{n=0}^{\infty} a_n \,(x - x_0)^n.$$

(b) Introduce the power series expansion above into the differential equation and find a *recurrence relation* among the coefficients a_n.

(c) Solve the recurrence relation in terms of free coefficients.

(d) If possible, add up the resulting power series for the solutions y_1, y_2.

We follow these steps in the examples below to find solutions to several differential equations. We start with a first order constant coefficient equation, and then we continue with a second order constant coefficient equation. The last two examples consider variable coefficient equations.

EXAMPLE 3.1.2: Find a power series solution y around the point $x_0 = 0$ of the equation

$$y' + c\, y = 0, \qquad c \in \mathbb{R}.$$

SOLUTION: We already know every solution to this equation. This is a first order, linear, differential equation, so using the method of integrating factor we find that the solution is

$$y(x) = a_0\, e^{-c\,x}, \qquad a_0 \in \mathbb{R}.$$

We are now interested in obtaining such solution with the power series method. Although this is not a second order equation, the power series method still works in this example. Propose a solution of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)}.$$

We can start the sum in y' at $n = 0$ or $n = 1$. We choose $n = 1$, since it is more convenient later on. Introduce the expressions above into the differential equation,

$$\sum_{n=1}^{\infty} n a_n\, x^{n-1} + c \sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Relabel the first sum above so that the functions x^{n-1} and x^n in the first and second sum have the same label. One way is the following,

$$\sum_{n=0}^{\infty} (n+1) a_{(n+1)}\, x^n + \sum_{n=0}^{\infty} c\, a_n\, x^n = 0$$

We can now write down both sums into one single sum,

$$\sum_{n=0}^{\infty} \big[(n+1) a_{(n+1)} + c\, a_n\big]\, x^n = 0.$$

Since the function on the left-hand side must be zero for every $x \in \mathbb{R}$, we conclude that every coefficient that multiplies x^n must vanish, that is,

$$(n+1) a_{(n+1)} + c\, a_n = 0, \qquad n \geqslant 0.$$

The last equation is called a *recurrence relation* among the coefficients a_n. The solution of this relation can be found by writing down the first few cases and then guessing the general expression for the solution, that is,

$$n = 0, \qquad a_1 = -c\, a_0 \qquad \Rightarrow \qquad a_1 = -c\, a_0,$$

$$n = 1, \qquad 2a_2 = -c\, a_1 \qquad \Rightarrow \qquad a_2 = \frac{c^2}{2!}\, a_0,$$

$$n = 2, \qquad 3a_3 = -c\, a_2 \qquad \Rightarrow \qquad a_3 = -\frac{c^3}{3!}\, a_0,$$

$$n = 3, \qquad 4a_4 = -c\, a_3 \qquad \Rightarrow \qquad a_4 = \frac{c^4}{4!}\, a_0.$$

One can check that the coefficient a_n can be written as

$$a_n = (-1)^n \frac{c^n}{n!}\, a_0,$$

which implies that the solution of the differential equation is given by

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{c^n}{n!}\, x^n \quad \Rightarrow \quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-c\, x)^n}{n!} \quad \Rightarrow \quad y(x) = a_0\, e^{-c\, x}.$$

◁

EXAMPLE 3.1.3: Find a power series solution $y(x)$ around the point $x_0 = 0$ of the equation

$$y'' + y = 0.$$

SOLUTION: We know that the solution can be found computing the roots of the characteristic polynomial $r^2 + 1 = 0$, which gives us the solutions

$$y(x) = a_0\, \cos(x) + a_1\, \sin(x).$$

We now recover this solution using the power series,

$$y = \sum_{n=0}^{\infty} a_n\, x^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n\, x^{(n-1)}, \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n\, x^{(n-2)}.$$

Introduce the expressions above into the differential equation, which involves only the function and its second derivative,

$$\sum_{n=2}^{\infty} n(n-1) a_n\, x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Relabel the first sum above, so that both sums have the same factor x^n. One way is,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)}\, x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Now we can write both sums using one single sum as follows,

$$\sum_{n=0}^{\infty} \big[(n+2)(n+1) a_{(n+2)} + a_n\big]\, x^n = 0 \quad \Rightarrow \quad (n+2)(n+1) a_{(n+2)} + a_n = 0. \quad n \geqslant 0.$$

The last equation is the *recurrence relation*. The solution of this relation can again be found by writing down the first few cases, and we start with even values of n, that is,

$$n = 0, \qquad (2)(1) a_2 = -a_0 \qquad \Rightarrow \qquad a_2 = -\frac{1}{2!}\, a_0,$$

$$n = 2, \qquad (4)(3) a_4 = -a_2 \qquad \Rightarrow \qquad a_4 = \frac{1}{4!}\, a_0,$$

$$n = 4, \qquad (6)(5) a_6 = -a_4 \qquad \Rightarrow \qquad a_6 = -\frac{1}{6!}\, a_0.$$

One can check that the even coefficients a_{2k} can be written as

$$a_{2k} = \frac{(-1)^k}{(2k)!}\, a_0.$$

The coefficients a_n for the odd values of n can be found in the same way, that is,

$$n = 1, \qquad (3)(2)a_3 = -a_1 \qquad \Rightarrow \qquad a_3 = -\frac{1}{3!}\, a_1,$$

$$n = 3, \qquad (5)(4)a_5 = -a_3 \qquad \Rightarrow \qquad a_5 = \frac{1}{5!}\, a_1,$$

$$n = 5, \qquad (7)(6)a_7 = -a_5 \qquad \Rightarrow \qquad a_7 = -\frac{1}{7!}\, a_1.$$

One can check that the odd coefficients a_{2k+1} can be written as

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)!}\, a_1.$$

Split the sum in the expression for y into even and odd sums. We have the expression for the even and odd coefficients. Therefore, the solution of the differential equation is given by

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}.$$

One can check that these are precisely the power series representations of the cosine and sine functions, respectively,

$$y(x) = a_0 \cos(x) + a_1 \sin(x). \qquad \triangleleft$$

Example 3.1.4: Find the first four terms of the power series expansion around the point $x_0 = 1$ of each fundamental solution to the differential equation

$$y'' - x\, y' - y = 0.$$

Solution: This is a differential equation we cannot solve with the methods of previous sections. This is a second order, variable coefficients equation. We use the power series method, so we look for solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad \Rightarrow \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \Rightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}.$$

We start working in the middle term in the differential equation. Since the power series is centered at $x_0 = 1$, it is convenient to re-write this term as $x\, y' = [(x-1)+1]\, y'$, that is,

$$\begin{aligned} x\, y' &= \sum_{n=1}^{\infty} n a_n x (x-1)^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n \big[(x-1)+1\big](x-1)^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n (x-1)^n + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}. \end{aligned} \tag{3.1.4}$$

As usual by now, the first sum on the right-hand side of Eq. (3.1.4) can start at $n = 0$, since we are only adding a zero term to the sum, that is,

$$\sum_{n=1}^{\infty} n a_n (x-1)^n = \sum_{n=0}^{\infty} n a_n (x-1)^n;$$

while it is convenient to relabel the second sum in Eq. (3.1.4) follows,

$$\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{(n+1)} (x-1)^n;$$

so both sums in Eq. (3.1.4) have the same factors $(x-1)^n$. We obtain the expression

$$\begin{aligned} x\,y' &= \sum_{n=0}^{\infty} n a_n (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{(n+1)} (x-1)^n \\ &= \sum_{n=0}^{\infty} \big[n a_n + (n+1) a_{(n+1)} \big] (x-1)^n. \end{aligned} \tag{3.1.5}$$

In a similar way relabel the index in the expression for y'', so we obtain

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-1)^n. \tag{3.1.6}$$

If we use Eqs. (3.1.5)-(3.1.6) in the differential equation, together with the expression for y, the differential equation can be written as follows

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-1)^n - \sum_{n=0}^{\infty} \big[n a_n + (n+1) a_{(n+1)} \big] (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0.$$

We can now put all the terms above into a single sum,

$$\sum_{n=0}^{\infty} \Big[(n+2)(n+1) a_{(n+2)} - (n+1) a_{(n+1)} - n a_n - a_n \Big] (x-1)^n = 0.$$

This expression provides the *recurrence relation* for the coefficients a_n with $n \geqslant 0$, that is,

$$\begin{aligned} (n+2)(n+1) a_{(n+2)} - (n+1) a_{(n+1)} - (n+1) a_n &= 0 \\ (n+1) \Big[(n+2) a_{(n+2)} - a_{(n+1)} - a_n \Big] &= 0, \end{aligned}$$

which can be rewritten as follows,

$$(n+2) a_{(n+2)} - a_{(n+1)} - a_n = 0. \tag{3.1.7}$$

We can solve this recurrence relation for the first four coefficients,

$$\begin{aligned} n &= 0 \qquad & 2a_2 - a_1 - a_0 &= 0 \qquad & \Rightarrow \qquad & a_2 = \frac{a_1}{2} + \frac{a_0}{2}, \\ n &= 1 \qquad & 3a_3 - a_2 - a_1 &= 0 \qquad & \Rightarrow \qquad & a_3 = \frac{a_1}{2} + \frac{a_0}{6}, \\ n &= 2 \qquad & 4a_4 - a_3 - a_2 &= 0 \qquad & \Rightarrow \qquad & a_4 = \frac{a_1}{4} + \frac{a_0}{6}. \end{aligned}$$

Therefore, the first terms in the power series expression for the solution y of the differential equation are given by

$$y = a_0 + a_1 (x-1) + \Big(\frac{a_0}{2} + \frac{a_1}{2} \Big) (x-1)^2 + \Big(\frac{a_0}{6} + \frac{a_1}{2} \Big) (x-1)^3 + \Big(\frac{a_0}{6} + \frac{a_1}{4} \Big) (x-1)^4 + \cdots$$

which can be rewritten as

$$\begin{aligned} y = \quad & a_0 \Big[1 + \frac{1}{2} (x-1)^2 + \frac{1}{6} (x-1)^3 + \frac{1}{6} (x-1)^4 + \cdots \Big] \\ &+ a_1 \Big[(x-1) + \frac{1}{2} (x-1)^2 + \frac{1}{2} (x-1)^3 + \frac{1}{4} (x-1)^4 + \cdots \Big] \end{aligned}$$

So the first four terms on each fundamental solution are given by

$$y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4,$$
$$y_2 = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4.$$

◁

Example 3.1.5: Find the first three terms of the power series expansion around the point $x_0 = 2$ of each fundamental solution to the differential equation

$$y'' - x\,y = 0.$$

Solution: We then look for solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n.$$

It is convenient to rewrite the function $x\,y = [(x-2)+2]\,y$, that is,

$$\begin{aligned} xy &= \sum_{n=0}^{\infty} a_n x (x-2)^n \\ &= \sum_{n=0}^{\infty} a_n \big[(x-2)+2\big](x-2)^n \\ &= \sum_{n=0}^{\infty} a_n (x-2)^{n+1} + \sum_{n=0}^{\infty} 2a_n (x-2)^n. \end{aligned} \tag{3.1.8}$$

We now relabel the first sum on the right-hand side of Eq. (3.1.8) in the following way,

$$\sum_{n=0}^{\infty} a_n (x-2)^{n+1} = \sum_{n=1}^{\infty} a_{(n-1)} (x-2)^n. \tag{3.1.9}$$

We do the same type of relabeling on the expression for y'',

$$\begin{aligned} y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-2)^n. \end{aligned}$$

Then, the differential equation above can be written as follows

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} (x-2)^n - \sum_{n=0}^{\infty} 2a_n (x-2)^n - \sum_{n=1}^{\infty} a_{(n-1)} (x-2)^n = 0$$
$$(2)(1)a_2 - 2a_0 + \sum_{n=1}^{\infty} \Big[(n+2)(n+1) a_{(n+2)} - 2a_n - a_{(n-1)}\Big] (x-2)^n = 0.$$

So the *recurrence relation* for the coefficients a_n is given by

$$a_2 - a_0 = 0, \qquad (n+2)(n+1) a_{(n+2)} - 2a_n - a_{(n-1)} = 0, \qquad n \geqslant 1.$$

We can solve this recurrence relation for the first four coefficients,

$$\begin{aligned}
n=0 &\qquad a_2 - a_0 = 0 &\Rightarrow&\qquad a_2 = a_0,\\
n=1 &\qquad (3)(2)a_3 - 2a_1 - a_0 = 0 &\Rightarrow&\qquad a_3 = \frac{a_0}{6} + \frac{a_1}{3},\\
n=2 &\qquad (4)(3)a_4 - 2a_2 - a_1 = 0 &\Rightarrow&\qquad a_4 = \frac{a_0}{6} + \frac{a_1}{12}.
\end{aligned}$$

Therefore, the first terms in the power series expression for the solution y of the differential equation are given by

$$y = a_0 + a_1(x-2) + a_0(x-2)^2 + \Big(\frac{a_0}{6} + \frac{a_1}{3}\Big)(x-2)^3 + \Big(\frac{a_0}{6} + \frac{a_1}{12}\Big)(x-2)^4 + \cdots$$

which can be rewritten as

$$\begin{aligned}
y = \;& a_0\Big[1 + (x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \cdots\Big]\\
&+ a_1\Big[(x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \cdots\Big]
\end{aligned}$$

So the first three terms on each fundamental solution are given by

$$y_1 = 1 + (x-2)^2 + \frac{1}{6}(x-2)^3,$$

$$y_2 = (x-2) + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4.$$

◁

3.1.3. **The Legendre Equation.** The Legendre equation appears when one solves the Laplace equation in spherical coordinates. The Laplace equation describes several phenomena, such as the static electric potential near a charged body, or the gravitational potential of a planet or star. When the Laplace equation describes a situation having spherical symmetry it makes sense to use spherical coordinates to solve the equation. It is in that case that the Legendre equation appears for a variable related to the polar angle in the spherical coordinate system. See Jackson's classic book on electrodynamics [8], § 3.1, for a derivation of the Legendre equation from the Laplace equation.

EXAMPLE 3.1.6: Find all solutions of the Legendre equation

$$(1-x^2)\,y'' - 2x\,y' + l(l+1)\,y = 0,$$

where l is any real constant, using power series centered at $x_0 = 0$.

SOLUTION: We start writing the equation in the form of Theorem 3.1.2,

$$y'' - \frac{2}{(1-x^2)}\,y' + \frac{l(l+1)}{(1-x^2)}\,y = 0.$$

It is clear that the coefficient functions

$$p(x) = -\frac{2}{(1-x^2)}, \qquad q(x) = \frac{l(l+1)}{(1-x^2)},$$

are analytic on the interval $|x| < 1$, which is centered at $x_0 = 0$. Theorem 3.1.2 says that there are two solutions linearly independent and analytic on that interval. So we write the solution as a power series centered at $x_0 = 0$,

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n,$$

Then we get,

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}\, x^n,$$

$$-x^2\, y'' = \sum_{n=0}^{\infty} -(n-1)na_n\, x^n,$$

$$-2x\, y' = \sum_{n=0}^{\infty} -2na_n\, x^n,$$

$$l(l+1)\, y = \sum_{n=0}^{\infty} l(l+1)a_n\, x^n.$$

The Legendre equation says that the addition of the four equations above must be zero,

$$\sum_{n=0}^{\infty} \big((n+2)(n+1)a_{(n+2)} - (n-1)na_n - 2na_n + l(l+1)a_n\big)\, x^n = 0.$$

Therefore,every term in that sum must vanish,

$$(n+2)(n+1)a_{(n+2)} - (n-1)na_n - 2na_n + l(l+1)a_n = 0, \qquad n \geqslant n.$$

This is the recurrence relation for the coefficients a_n. After a few manipulations the recurrence relation becomes

$$a_{(n+2)} = -\frac{(l-n)(l+n+1)}{(n+2)(n+1)}\, a_n, \qquad n \geqslant 0.$$

By giving values to n we obtain,

$$a_2 = -\frac{l(l+1)}{2!}\, a_0, \quad a_3 = -\frac{(l-1)(l+2)}{3!}\, a_1.$$

Since a_4 is related to a_2 and a_5 is related to a_3, we get,

$$a_4 = -\frac{(l-2)(l+3)}{(3)(4)}\, a_2 \quad \Rightarrow \quad a_4 = \frac{(l-2)l(l+1)(l+3)}{4!}\, a_0,$$

$$a_5 = -\frac{(l-3)(l+4)}{(4)(5)}\, a_3 \quad \Rightarrow \quad a_5 = \frac{(l-3)(l-1)(l+2)(l+4)}{5!}\, a_1.$$

If one keeps solving the coefficients a_n in terms of either a_0 or a_1, one gets the expression,

$$\begin{aligned} y(x) = \quad & a_0 \Big[1 - \frac{l(l+1)}{2!}\, x^2 + \frac{(l-2)l(l+1)(l+3)}{4!}\, x^4 + \cdots\Big] \\ & + a_1 \Big[x - \frac{(l-1)(l+2)}{3!}\, x^3 + \frac{(l-3)(l-1)(l+2)(l+4)}{5!}\, x^5 + \cdots\Big]. \end{aligned}$$

Hence, the fundamental solutions are

$$y_1(x) = 1 - \frac{l(l+1)}{2!}\, x^2 + \frac{(l-2)l(l+1)(l+3)}{4!}\, x^4 + \cdots$$

$$y_2(x) = x - \frac{(l-1)(l+2)}{3!}\, x^3 + \frac{(l-3)(l-1)(l+2)(l+4)}{5!}\, x^5 + \cdots.$$

The ration test provides the interval where the seires above converge. For function y_1 we get, replacing n by $2n$,

$$\Big|\frac{a_{2n+2}\, x^{2n+2}}{a_{2n}\, x^{2n}}\Big| = \Big| = \frac{(l-2n)(l+2n+1)}{(2n+1)(2n+2)}\Big|\, |x^2| \to |x|^2 \quad \text{as} \quad n \to \infty.$$

A similar result holds for y_2. So both series converge on the interval defined by $|x| < 1$. ◁

Remark: The functions y_1, y_2 are called Legendre functions. For a noninteger value of the constant l these functions cannot be written in terms of elementary functions. But when l is an integer, one of these series terminate and becomes a polynomial. The case l being a nonnegative integer is specially relevant in physics. For l even the function y_1 becomes a polynomial while y_2 remains an infinite series. For l odd the function y_2 becomes a polynomial while the y_1 remains an infinite series. For example, for $l = 0, 1, 2, 3$ we get,

$$\begin{aligned} l &= 0, & y_1(x) &= 1, \\ l &= 1, & y_2(x) &= x, \\ l &= 2, & y_1(x) &= 1 - 3x^2, \\ l &= 3, & y_2(x) &= x - \frac{5}{3}\,x^3. \end{aligned}$$

The Legendre polynomials are proportional to these polynomials. The proportionality factor for each polynomial is chosen so that the Legendre polynomials have unit lengh in a particular chosen inner product. We just say here that the first four polynomials are

$$\begin{aligned} l &= 0, & y_1(x) &= 1, & P_0 &= y_1, & P_0(x) &= 1, \\ l &= 1, & y_2(x) &= x, & P_1 &= y_2, & P_1(x) &= x, \\ l &= 2, & y_1(x) &= 1 - 3x^2, & P_2 &= -\frac{1}{2}\,y_1, & P_2(x) &= \frac{1}{2}\big(3x^2 - 1\big), \\ l &= 3, & y_2(x) &= x - \frac{5}{3}\,x^3, & P_3 &= -\frac{3}{2}\,y_1, & P_3(x) &= \frac{1}{2}\big(5x^3 - 3x\big). \end{aligned}$$

These polynomials, P_n, are called Legendre polynomials. The graph of the first four Legendre polynomials is given in Fig. 14.

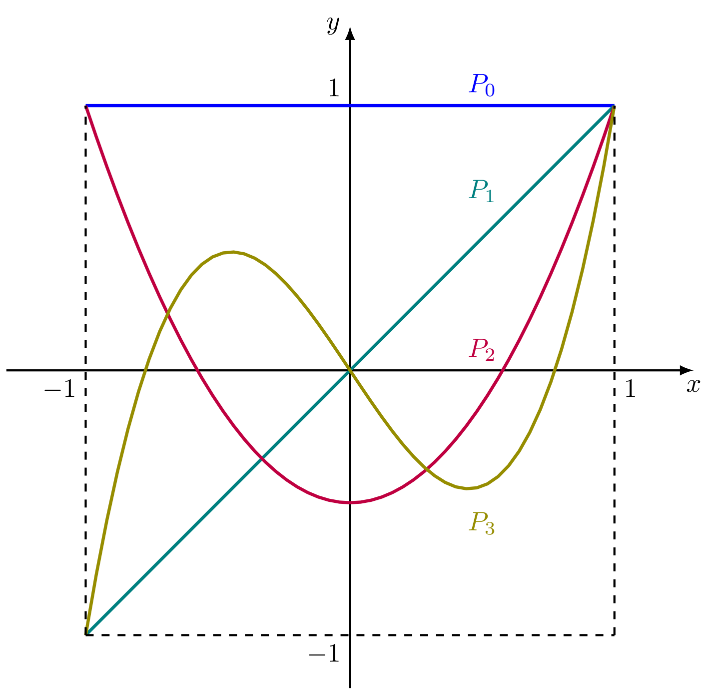


FIGURE 14. The graph of the first four Legendre polynomials.

3.1.4. Exercises.

3.1.1.- .

3.1.2.- .

3.2. The Euler Equidimensional Equation

When the coefficients p and q are analytic functions on a given domain, the equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

has analytic fundamental solutions on that domain. This is the main result in § 3.1, Theorem 3.1.2. Recall that a function is analytic on an open domain iff the function admits a convergent power series on that domain. In this section we start the study of equations where the coefficients p and q are not analytic functions. We want to study equations with coefficients p and q having singularities. We want to find solutions defined arbitrarily close to these singularities. On the one hand, this is an important problem because many differential equations in physics have coefficients with singularities. And finding solutions with physical meaning often involves studying all solutions near these singularities. On the other hand, this is a difficult problem, and when p and q are completely arbitrary functions there is not much that can be learned from the solutions of the equation. For this reason we start our study with one of the simplest cases, Euler's equidimensional equation.

3.2.1. The Roots of the Indicial Polynomial.

We study the differential equation

$$y'' + p(x)\, y' + q(x)\, y = 0,$$

where the coefficients p and q are given by

$$p(x) = \frac{p_0}{(x - x_0)}, \qquad q(x) = \frac{q_0}{(x - x_0)^2},$$

with p_0 and q_0 constants. The point x_0 is a singular point of the equation; the functions p and q are not analytic on an open set including x_0. But the singularity is of a good type, the type we know how find solutions. We start with a small rewriting of the differential equation we are going to study.

Definition 3.2.1. *The* ***Euler equidimensional equation*** *for the unknown y with singular point at $x_0 \in \mathbb{R}$ is given by the equation below, where p_0 and q_0 are constants,*

$$(x - x_0)^2\, y'' + p_0\, (x - x_0)\, y' + q_0\, y = 0.$$

Remarks:

(a) This equation is also called Cauchy equidimensional equation, Cauchy equation, Cauchy-Euler equation, or simply Euler equation. As George Simmons says in [10], "Euler studies were so extensive that many mathematicians tried to avoid confusion by naming subjects after the person who first studied them after Euler."

(b) The equation is called equidimensional because if the variable x has any physical dimensions, then the terms with $(x - x_0)^n \dfrac{d^n}{dx^n}$, for any nonnegative integer n, are actually dimensionless.

(c) The exponential functions $y(x) = e^{rx}$ are not solutions of the Euler equation. Just introduce such a function into the equation, and it is simple to show that there is no constant r such that the exponential is solution.

(d) As we mentioneed above, the point $x_0 \in \mathbb{R}$ is a *singular point* of the equation.

(e) The particular case $x_0 = 0$ is

$$x^2\, y'' + p_0\, x\, y' + q_0\, y = 0.$$

We now summarize what is known about solutions of the Euler equation.

Theorem 3.2.2 (Euler Equation). *Consider the Euler equidimensional equation*

$$(x - x_0)^2\, y'' + p_0\,(x - x_0)\, y' + q_0\, y = 0, \qquad x > x_0, \tag{3.2.1}$$

where p_0, q_0, and x_0 are real constants, and let $r_\pm$ be the roots of the indicial polynomial $p(r) = r(r-1) + p_0 r + q_0$.

(a) If $r_+ \neq r_-$, real or complex, then the general solution of Eq. (3.2.1) is given by

$$y_{gen}(t) = c_+(x - x_0)^{r_+} + c_-(x - x_0)^{r_-}, \quad x > x_0, \quad c_+,\ c_- \in \mathbb{R}.$$

(b) If $r_+ = r_- = r_0 \in \mathbb{R}$, then the general solution of Eq. (3.2.1) is given by

$$y_{gen}(t) = c_+\,(x - x_0)^{r_0} + c_-\,(x - x_0)^{r_0}\,\ln(x - x_0), \quad x > x_0, \quad c_+,\ c_- \in \mathbb{R}.$$

Furthermore, given real constants $x_1 \neq x_0$, y_0 and y_1, there is a unique solution to the initial value problem given by Eq. (3.2.1) and the initial conditions

$$y(x_1) = y_0, \qquad y'(x_1) = y_1.$$

Remark: We have restricted to a domain with $x > x_0$. Similar results hold for $x < x_0$. In fact one can prove the following: If a solution y has the value $y(x - x_0)$ at $x - x_0 > 0$, then the function $\tilde{y}$ defined as $\tilde{y}(x - x_0) = y(-(x - x_0))$, for $x - x_0 < 0$ is solution of Eq. (3.2.1) for $x - x_0 < 0$. For this reason the solution for $x \neq x_0$ is sometimes written in the literature, see [3] § 5.4, as follows,

$$y_{\text{gen}}(t) = c_+|x - x_0|^{r_+} + c_-|x - x_0|^{r_-}, \quad r_+ \neq r_-,$$
$$y_{\text{gen}}(t) = c_+\,|x - x_0|^{r_0} + c_-\,|x - x_0|^{r_0}\,\ln|x - x_0|, \quad r_+ = r_- = r_0.$$

However, when solving an initial value problem, we need to pick the domain that contains the initial data point x_1. This domain will be a subinterval in either $(-\infty, x_0)$ or (x_0, ∞).

The proof of this theorem closely follows the ideas to find all solutions of second order linear equations with constant coefficients, Theorem 2.3.2, in § 2.3. We first found fundamental solutions to the differential equation

$$y'' + a_1\, y' + a_0\, y = 0,$$

and then we recalled that Theorem 2.1.7 says that any other solution is a linear combinantion of any fundamental solutions pair. To get fundamental solutions we looked for exponential functions $y(x) = e^{rx}$, where the constant r was a root of the characteristic polynomial

$$r^2 + a_1 r + a_0 = 0.$$

When this polynomial had two different roots, $r_+ \neq r_-$, we got the fundamental solutions

$$y_+(x) = e^{r_+ x}, \qquad y_-(x) = e^{r_- x}.$$

When the root was repeated, $r_+ = r_- = r_0$, we used the reduction order method to get the fundamental solutions

$$y_+(x) = e^{r_0 x}, \qquad y_-(x) = x\, e^{r_0 x}.$$

Well, the proof of Theorem 3.2.2 closely follows this proof, replacing the exponential function by power functions.

Proof of Theorem 3.2.2: For simplicity we consider the case $x_0 = 0$. The general case $x_0 \neq 0$ follows from the case $x_0 = 0$ replacing x by $(x - x_0)$. So, consider the equation

$$x^2\, y'' + p_0\, x\, y' + q_0\, y = 0, \qquad x > 0.$$

We look for solutions of the form $y(x) = x^r$, because power functions have the property that

$$y' = r\, x^{r-1} \quad \Rightarrow \quad x\, y' = r\, x^r.$$

A similar property holds for the second derivative,

$$y'' = r(r-1)\,x^{r-2} \quad \Rightarrow \quad x^2\,y'' = r(r-1)\,x^r.$$

When we introduce this function into the Euler equation we get an algebraic equation for r,

$$\big[r(r-1) + p_0 r + q_0\big]\,x^r = 0 \quad \Leftrightarrow \quad r(r-1) + p_0 r + q_0 = 0.$$

The constant r must be a root of the indicial polynomial

$$p(r) = r(r-1) + p_0 r + q_0.$$

This polynomial is sometimes called the Euler characteristic polynomial. So we have two possibilities. If the roots are different, $r_+ \neq r_-$, we get the fundamental solutions

$$y_+(x) = x^{r_+}, \qquad y_-(x) = x^{r_-}.$$

If we have a repeated root $r_+ = r_- = r_0$, thhen one solution is $y_+(x) = x^{r_0}$. To obtain the second solution we use the reduction order method. Since we have one solution to the equation, y_+, the second solution is

$$y_-(x) = v(x)\,y_+(x) \quad \Rightarrow \quad y_-(x) = v(x)\,x^{r_0}.$$

We need to compute the first two derivatives of y_-,

$$y_-' = r_0 v\,x^{r_0-1} + v'\,x^{r_0}, \qquad y_-'' = r_0(r_0-1)v\,x^{r_0-2} + 2r_0 v'\,x^{r_0-1} + v''\,x^{r_0}.$$

We now put these expressions for y_-, y_-' and y_-'' into the Euler equation,

$$x^2\left(r_0(r_0-1)v\,x^{r_0-2} + 2r_0 v'\,x^{r_0-1} + v''\,x^{r_0}\right) + p_0 x\left(r_0 v\,x^{r_0-1} + v'\,x^{r_0}\right) + q_0\,v\,x^{r_0} = 0.$$

Let us reorder terms in the following way,

$$v''\,x^{r_0+2} + (2r_0 + p_0)\,v'\,x^{r_0+1} + \big[r_0(r_0-1) + p_0 r_0 + q_0\big]\,v\,x^{r_0} = 0.$$

We now need to recall both that r_0 is a root of the indicial polynomial,

$$r_0(r_0-1) + p_0 r_0 + q_0 = 0,$$

and that r_0 is a repeated root, that is $(p_0-1)^2 = 4q_0$, hence

$$r_0 = -\frac{(p_0-1)}{2} \quad \Rightarrow \quad 2r_0 + p_0 = 1.$$

Using these two properties of r_0 in the Euler equation above, we get the equation for v,

$$v''\,x^{r_0+2} + v'\,x^{r_0+1} = 0 \quad \Rightarrow \quad v''\,x + v' = 0.$$

This is a first order equation for $w = v'$,

$$w'\,x + w = 0 \quad \Rightarrow \quad (x\,w)' = 0 \quad \Rightarrow \quad w(x) = \frac{w_0}{x}.$$

We now integrate one last time to get function v,

$$v' = \frac{w_0}{x} \quad \Rightarrow \quad v(x) = w_0 \ln(x) + v_0.$$

So the second solution to the Euler equation in the case of repeated roots is

$$y_-(x) = \big(w_0 \ln(x) + v_0\big)\,x^{r_0} \quad \Rightarrow \quad y_-(x) = w_0 x^{r_0} \ln(x) + v_0\,y_+(x).$$

It is clear we can choose $v_0 = 0$ and $w_0 = 1$ to get

$$y_-(x) = x^{r_0} \ln(x).$$

We have found fundamental solutions for all possible roots of the indicial polynomial. The formulas for the general solutions follow from Theorem 2.1.7 in § 2.1. This establishes the Theorem. □

EXAMPLE 3.2.1: Find the general solution of the Euler equation below for $x > 0$,

$$x^2\, y'' + 4x\, y' + 2\, y = 0.$$

SOLUTION: We look for solutions of the form $y(x) = x^r$, which implies that

$$x\, y'(x) = r x^r, \qquad x^2\, y''(x) = r(r-1)\, x^r,$$

therefore, introducing this function y into the differential equation we obtain

$$\big[r(r-1) + 4r + 2\big]\, x^r = 0 \quad \Leftrightarrow \quad r(r-1) + 4r + 2 = 0.$$

The solutions are computed in the usual way,

$$r^2 + 3r + 2 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[-3 \pm \sqrt{9-8}\big] \quad \Rightarrow \quad \begin{cases} r_+ = -1 \\ r_- = -2. \end{cases}$$

So the general solution of the differential equation above is given by

$$y_{\text{gen}}(x) = c_+\, x^{-1} + c_-\, x^{-2}.$$

◁

Remark: Both fundamental solutions in the example above are not analytic on any interval including $x = 0$. Both solutions diverge at $x = 0$.

EXAMPLE 3.2.2: Find the general solution of the Euler equation below for $x > 0$,

$$x^2\, y'' - 3x\, y' + 4\, y = 0.$$

SOLUTION: We look for solutions of the form $y(x) = x^r$, then the constant r must be solution of the Euler characteristic polynomial

$$r(r-1) - 3r + 4 = 0 \quad \Leftrightarrow \quad r^2 - 4r + 4 = 0 \quad \Rightarrow \quad r_+ = r_- = 2.$$

Therefore, the general solution of the Euler equation in this case is given by

$$y_{\text{gen}}(x) = c_+ x^2 + c_- x^2 \ln(x).$$

◁

Remark: The fundamental solution $y_+(x) = x^2$ is analytic at $x = 0$, but the solution $y_-(x) = x^2 \ln(x)$ is not.

EXAMPLE 3.2.3: Find the general solution of the Euler equation below for $x > 0$,

$$x^2\, y'' - 3x\, y' + 13\, y = 0.$$

SOLUTION: We look for solutions of the form $y(x) = x^r$, which implies that

$$x\, y'(x) = r x^r, \qquad x^2\, y''(x) = r(r-1)\, x^r,$$

therefore, introducing this function y into the differential equation we obtain

$$\big[r(r-1) - 3r + 13\big]\, x^r = 0 \quad \Leftrightarrow \quad r(r-1) - 3r + 13 = 0.$$

The solutions are computed in the usual way,

$$r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{-36}\big] \quad \Rightarrow \quad \begin{cases} r_+ = 2 + 3i \\ r_- = 2 - 3i. \end{cases}$$

So the general solution of the differential equation above is given by

$$y_{\text{gen}}(x) = c_+\, x^{(2+3i)} + c_-\, x^{(2-3i)}. \tag{3.2.2}$$

◁

3.2.2. **Real Solutions for Complex Roots.** We study in more detail the solutions to the Euler equation in the case that the indicial polynomial has complex roots. Since these roots have the form

$$r_{\pm} = -\frac{(p_0 - 1)}{2} \pm \frac{1}{2}\sqrt{(p_0 - 1)^2 - 4q_0},$$

the roots are complex-valued in the case $(p_0 - 1)^2 - 4q_0 < 0$. We use the notation

$$r_{\pm} = \alpha \pm i\beta, \quad \text{with} \quad \alpha = -\frac{(p_0 - 1)}{2}, \quad \beta = \sqrt{q_0 - \frac{(p_0 - 1)^2}{4}}.$$

The fundamental solutions in Theorem 3.2.2 are the complex-valued functions

$$\tilde{y}_{+}(x) = x^{(\alpha + i\beta)}, \qquad \tilde{y}_{-}(x) = x^{(\alpha - i\beta)}.$$

The general solution constructed from these solutions is

$$y_{\text{gen}}(x) = \tilde{c}_{+}\, x^{(\alpha + i\beta)} + \tilde{c}_{-}\, x^{(\alpha - i\beta)}, \qquad \tilde{c}_{+},\ \tilde{c}_{-} \in \mathbb{C}.$$

This formula for the general solution includes real valued and complex valued solutions. But it is not so simple to single out the real valued solutions. Knowing the real valued solutions could be important in physical applications. If a physical system is described by a differential equation with real coefficients, more often than not one is interested in finding real valued solutions. For that reason we now provide a new set of fundamental solutions that are real valued. Using real valued fundamental solution is simple to separate all real valued solutions from the complex valued ones.

Theorem 3.2.3 (Real Valued Fundamental Solutions). *If the differential equation*

$$(x - x_0)^2\, y'' + p_0(x - x_0)\, y' + q_0\, y = 0, \qquad x > x_0, \tag{3.2.3}$$

where p_0, q_0, x_0 are real constants, has indicial polynomial with complex roots $r_{\pm} = \alpha \pm i\beta$ and complex valued fundamental solutions for $x > x_0$,

$$\tilde{y}_{+}(x) = (x - x_0)^{(\alpha + i\beta)}, \qquad \tilde{y}_{-}(x) = (x - x_0)^{(\alpha - i\beta)},$$

then the equation also has real valued fundamental solutions for $x > x_0$ given by

$$y_{+}(x) = (x - x_0)^{\alpha} \cos\big(\beta \ln(x - x_0)\big), \qquad y_{-}(x) = (x - x_0)^{\alpha} \sin\big(\beta \ln(x - x_0)\big).$$

Proof of Theorem 3.2.3: For simplicity consider the case $x_0 = 0$. Take the solutions

$$\tilde{y}_{+}(x) = x^{(\alpha + i\beta)}, \qquad \tilde{y}_{-}(x) = x^{(\alpha - i\beta)}.$$

Rewrite the power function as follows,

$$\tilde{y}_{+}(x) = x^{(\alpha + i\beta)} = x^{\alpha}\, x^{i\beta} = x^{\alpha}\, e^{\ln(x^{i\beta})} = x^{\alpha}\, e^{i\beta \ln(x)} \quad \Rightarrow \quad \tilde{y}_{+}(x) = x^{\alpha}\, e^{i\beta \ln(x)}.$$

A similar calculation yields

$$\tilde{y}_{-}(x) = x^{\alpha}\, e^{-i\beta \ln(x)}.$$

Recall now Euler formula for complex exponentials, $e^{i\theta} = \cos(\theta) + i \sin(\theta)$, then we get

$$\tilde{y}_{+}(x) = x^{\alpha}\, \big[\cos\big(\beta \ln(x)\big) + i \sin\big(\beta \ln(x)\big)\big], \qquad \tilde{y}_{-}(x) = x^{\alpha}\, \big[\cos\big(\beta \ln(x)\big) - i \sin\big(\beta \ln(x)\big)\big].$$

Since $\tilde{y}_{+}$ and $\tilde{y}_{-}$ are ssolutions to Eq. (3.2.3), so are the functions

$$y_1(x) = \frac{1}{2}\big[\tilde{y}_1(x) + \tilde{y}_2(x)\big], \qquad y_2(x) = \frac{1}{2i}\big[\tilde{y}_1(x) - \tilde{y}_2(x)\big].$$

It is not difficult to see that these functions are

$$y_{+}(x) = x^{\alpha} \cos\big(\beta \ln(x)\big), \qquad y_{-}(x) = x^{\alpha} \sin\big(\beta \ln(x)\big).$$

To prove the case having $x_0 \neq 0$, just replace x by $(x - x_0)$ on all steps above. This establishes the Theorem. □

EXAMPLE 3.2.4: Find a real-valued general solution of the Euler equation below for $x > 0$,

$$x^2\, y'' - 3x\, y' + 13\, y = 0.$$

SOLUTION: The indicial equation is $r(r-1) - 3r + 13 = 0$, with solutions

$$r^2 - 4r + 13 = 0 \quad \Rightarrow \quad r_+ = 2 + 3i, \quad r_- = 2 - 3i.$$

A complex-valued general solution for $x > 0$ is,

$$y_{\text{gen}}(x) = \tilde{c}_+\, x^{(2+3i)} + \tilde{c}_-\, x^{(2-3i)} \quad \tilde{c}_+,\ \tilde{c}_- \in \mathbb{C}.$$

A real-valued general solution for $x > 0$ is

$$y_{\text{gen}}(x) = c_+\, x^2 \cos\big(3\ln(x)\big) + c_-\, x^2 \sin\big(3\ln(x)\big), \qquad c_+,\ c_- \in \mathbb{R}.$$

$\lhd$

3.2.3. **Transformation to Constant Coefficients.** Teorem 3.2.2 shows that power functions $y(x) = x^{r_\pm}$, where $r_\pm$ the roots of the indicial polynomial, are solutions to the Euler equidimensional equation

$$x^2\, y'' + p_0 x\, y' + q_0\, y = 0, \qquad x > 0.$$

The proof of this theorem is to verify that the power functions $y(x) = x^{r_\pm}$ solve the differential equation. How did we know we had to try with power functions? One answer could be, this is a guess, a lucky one. Another answer could be that the Euler equation can be transformed into a constant coefficient equation by a change of variable.

Theorem 3.2.4. *The function y is solution of the Euler equidimensional equation*

$$x^2\, y'' + p_0 x\, y' + q_0\, y = 0, \qquad x > 0$$

iff the function $u(z) = y(e^z)$ satisfies the constant coefficients equation

$$\ddot{u} + (p_0 - 1)\, \dot{u} + q_0\, u = 0,$$

where $y' = dy/dx$ and $\dot{u} = du/dz$. Furthermore, the functions $y(x) = e^{r_\pm x}$ are solutions of the Euler equidimensional equation iff the constants $r_\pm$ are solutions of the indicial equation

$$r_\pm^2 + (p_0 - 1) r_\pm + q_0 = 0.$$

Proof of Theorem 3.2.4: Given $x > 0$, introduce $z(x) = \ln(x)$, therefore $x(z)) = e^z$. Given a function y, introduce the function

$$u(z) = y(x(z)) \quad \Rightarrow \quad u(z) = y(e^z).$$

Then, the derivatives of u and y are related by the chain rule,

$$\dot{u}(z) = \frac{du}{dz}(z) = \frac{dy}{dx}(x(z))\, \frac{dx}{dz}(z) = y'(x(z))\, \frac{d(e^z)}{dz} = y'(x(z))\, e^z$$

so we obtain

$$\dot{u}(z) = x\, y'(x),$$

where we have denoted $\dot{u} = du/dz$. The relation for the second derivatives is

$$\ddot{u}(z) = \frac{d}{dx}\big(x\, y'(x)\big)\, \frac{dx}{dz}(z) = \big(x\, y''(x) + y'(x)\big)\, \frac{d(e^z)}{dz} = \big(x\, y''(x) + y'(x)\big)\, x$$

so we obtain

$$\ddot{u}(z) = x^2\, y''(x) + x\, y'(x).$$

Combining the equations for $\dot{u}$ and $\ddot{u}$ we get

$$x^2\, y'' = \ddot{u} - \dot{u}, \qquad x\, y' = \dot{u}.$$

The function y is solution of the Euler equation $x^2\, y'' + p_0 x\, y' + q_0\, y = 0$ iff holds

$$\ddot{u} - \dot{u} + p_0\, \dot{u} + q_0\, u = 0 \quad \Rightarrow \quad \ddot{u} + (p_0 - 1)\, \dot{u} + q_0\, u = 0.$$

This is a second order linear equation with constant coefficients. The solutions are

$$u(z) = e^{r_\pm z}, \qquad r_\pm^2 + (p_0 - 1) r_\pm + q_0 = 0.$$

So $r_\pm$ must be a root of the *indicial polynomial*. Recalling that $y(x) = u(z(x))$, we get

$$y(x) = e^{r_\pm\, z(x)} = e^{r_\pm \ln(x)} = e^{\ln(x^{r_\pm})} \quad \Rightarrow \quad y(x) = x^{r_\pm}.$$

This establishes the Theorem. □

3.2.4. **Exercises.**

3.2.1.- .

3.2.2.- .

3.3. Solutions Near Regular Singular Points

We continue with our study of the solutions to the differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

In § 3.1 we studied the case where the coefficient functions p and q were analytic functions. We saw that the solutions were also analytic and we used power series to find them. In § 3.2 we studied the case where the coefficients p and q were singular at a point x_0. The singularity was of a very particular form,

$$p(x) = \frac{p_0}{(x - x_0)}, \qquad q(x) = \frac{q_0}{(x - x_0)^2},$$

where p_0, q_0 are constants. The equation was called the Euler equidimensional equation. We found solutions near the singular point x_0. We found out that some solutions were analytic at x_0 and some solutions were singular at x_0. In this section we study equations with coefficients p and q being again singular at a point x_0. The singularity in this case is such that both functions below

$$(x - x_0)p(x), \qquad (x - x_0)^2 q(x)$$

are analytic in a neighborhood of x_0. The Euler equation is the particular case where these functions above are constants. Now we say they admit power series expansions centered at x_0. So we study equations that are close to Euler equations when the variable x is close to the singular point x_0. We will call the point x_0 a regular singular point. That is, a singular point that is not so singular. We will find out that some solutions may be well defined at the regular singular point and some other solutions may be singular at that point.

3.3.1. **Regular Singular Points.** In § 3.1 we studied second order equations

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

and we looked for solutions near regular points of the equation. A point x_0 is a regular point of the equation iff the functions p and q are analytic in a neighborhood of x_0. In particular the definition means that these functions have power series centered at x_0,

$$p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n,$$

which converge in a neighborhood of x_0. A point x_0 is called a singular point of the equation if the coefficients p and q are not analytic on any set containing x_0. In this section we study a particular type of singular points. We study singular points that are not so singular.

Definition 3.3.1. *A point $x_0 \in \mathbb{R}$ is a* ***regular singular point*** *of the equation*

$$y'' + p(x)\,y' + q(x)\,y = 0.$$

iff both functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ are analytic on a neighborhood containing x_0, where

$$\tilde{p}_{x_0}(x) = (x - x_0)p(x), \qquad \tilde{q}_{x_0}(x) = (x - x_0)^2 q(x).$$

Remark: The singular point x_0 in an Euler equidimensional equation is regular singular. In fact, the functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ are not only analytic, they are actually constant. The proof is simple, take the Euler equidimensional equation

$$y'' + \frac{p_0}{(x - x_0)}\,y' + \frac{q_0}{(x - x_0)^2}\,y = 0,$$

and compute the functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ for the point x_0,

$$\tilde{p}_{x_0}(x) = (x - x_0)\Big(\frac{p_0}{(x - x_0)}\Big) = p_0, \qquad \tilde{q}_{x_0}(x) = (x - x_0)^2\Big(\frac{q_0}{(x - x_0)^2}\Big) = q_0.$$

EXAMPLE 3.3.1: Show that the singular point of Euler equation below is regular singular,

$$(x-3)^2\, y'' + 2(x-3)\, y' + 4\, y = 0.$$

SOLUTION: Divide the equation by $(x-3)^2$, so we get the equation in the standard form

$$y'' + \frac{2}{(x-3)}\, y' + \frac{4}{(x-3)^2}\, y = 0.$$

The functions p and q are given by

$$p(x) = \frac{2}{(x-3)}, \qquad q(x) = \frac{4}{(x-3)^2}.$$

The functions $\tilde{p}_3$ and $\tilde{q}_3$ for the point $x_0 = 3$ are constants,

$$\tilde{p}_3(x) = (x-3)\Big(\frac{2}{(x-3)}\Big) = 2, \qquad \tilde{q}_3(x) = (x-3)^2\Big(\frac{4}{(x-3)^2}\Big) = 4.$$

Therefore they are analytic. This shows that $x_0 = 3$ is regular singular. ◁

EXAMPLE 3.3.2: Find the regular-singular points of the Legendre equation

$$(1-x^2)\, y'' - 2x\, y' + l(l+1)\, y = 0,$$

where l is a real constant.

SOLUTION: We start writing the Legendre equation in the standard form

$$y'' - \frac{2x}{(1-x^2)}\, y' + \frac{l(l+1)}{(1-x^2)}\, y = 0,$$

The functions p and q are given by

$$p(x) = -\frac{2x}{(1-x^2)}, \qquad q(x) = \frac{l(l+1)}{(1-x^2)}.$$

These functions are analytic except where the denominators vanish.

$$(1-x^2) = (1-x)(1+x) = 0 \quad \Rightarrow \quad \begin{cases} x_0 = 1, \\ x_1 = -1. \end{cases}$$

Let us start with the singular point $x_0 = 1$. The functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ for this point are,

$$\tilde{p}_{x_0}(x) = (x-1)p(x) = (x-1)\Big(-\frac{2x}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{p}_{x_0}(x) = \frac{2x}{(1+x)},$$

$$\tilde{q}_{x_0}(x) = (x-1)^2 q(x) = (x-1)^2\Big(\frac{l(l+1)}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_0}(x) = -\frac{l(l+1)(x-1)}{(1+x)}.$$

These two functions are analytic in a neighborhood of $x_0 = 1$. (Both $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ have no vertical asymptote at $x_0 = 1$.) Therefore, the point $x_0 = 1$ is a regular singular point. We now need to do a similar calculation with the point $x_1 = -1$. The functions $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ for this point are,

$$\tilde{p}_{x_1}(x) = (x+1)p(x) = (x+1)\Big(-\frac{2x}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{p}_{x_1}(x) = -\frac{2x}{(1-x)},$$

$$\tilde{q}_{x_1}(x) = (x+1)^2 q(x) = (x+1)^2\Big(\frac{l(l+1)}{(1-x)(1+x)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_1}(x) = \frac{l(l+1)(x+1)}{(1-x)}.$$

These two functions are analytic in a neighborhood of $x_1 = -1$. (Both $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ have no vertical asymptote at $x_1 = -1$.) Therefore, the point $x_1 = -1$ is a regular singular point. ◁

EXAMPLE 3.3.3: Find the regular singular points of the differential equation

$$(x+2)^2(x-1)\,y'' + 3(x-1)\,y' + 2\,y = 0.$$

SOLUTION: We start writing the equation in the standard form

$$y'' + \frac{3}{(x+2)^2}\,y' + \frac{2}{(x+2)^2(x-1)}\,y = 0.$$

The functions p and q are given by

$$p(x) = \frac{3}{(x+2)^2}, \qquad q(x) = \frac{2}{(x+2)^2(x-1)}.$$

The denominators of the functions above vanish at $x_0 = -2$ and $x_1 = 1$. These are singular points of the equation. Let us find out whether these singular points are regular singular or not. Let us start with $x_0 = -2$. The functions $\tilde{p}_{x_0}$ and $\tilde{q}_{x_0}$ for this point are,

$$\tilde{p}_{x_0}(x) = (x+2)p(x) = (x+2)\Big(\frac{3}{(x+2)^2}\Big) \quad \Rightarrow \quad \tilde{p}_{x_0}(x) = \frac{3}{(x+2)},$$

$$\tilde{q}_{x_0}(x) = (x+2)^2 q(x) = (x+2)^2\Big(\frac{2}{(x+2)^2(x-1)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_0}(x) = -\frac{2}{(x-1)}.$$

We see that $\tilde{q}_{x_0}$ is analytic on a neighborhood of $x_0 = -2$, but $\tilde{p}_{x_0}$ is not analytic on any neighborhood containing $x_0 = -2$, because the function $\tilde{p}_{x_0}$ has a vertical asymptote at $x_0 = -2$. So the point $x_0 = -2$ is not a regular singular point. We need to do a similar calculation for the singular point $x_1 = 1$. The functions $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ for this point are,

$$\tilde{p}_{x_1}(x) = (x-1)p(x) = (x-1)\Big(\frac{3}{(x+2)^2}\Big) \quad \Rightarrow \quad \tilde{p}_{x_1}(x) = \frac{3(x-1)}{(x+2)},$$

$$\tilde{q}_{x_1}(x) = (x-1)^2 q(x) = (x-1)^2\Big(\frac{2}{(x+2)^2(x-1)}\Big) \quad \Rightarrow \quad \tilde{q}_{x_1}(x) = -\frac{2(x-1)}{(x+2)}.$$

We see that both functions $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ are analytic on a neighborhood containing $x_1 = 1$. (Both $\tilde{p}_{x_1}$ and $\tilde{q}_{x_1}$ have no vertical asymptote at $x_1 = 1$.) Therefore, the point $x_1 = 1$ is a regular singular point. $\lhd$

Remark: It is fairly simple to find the regular singular points of an equation. Take the equation in out last example, written in standard form,

$$y'' + \frac{3}{(x+2)^2}\,y' + \frac{2}{(x+2)^2(x-1)}\,y = 0.$$

The functions p and q are given by

$$p(x) = \frac{3}{(x+2)^2}, \qquad q(x) = \frac{2}{(x+2)^2(x-1)}.$$

The singular points are given by the zeros in the denominators, that is $x_0 = -2$ and $x_1 = 1$. The point x_0 is not regular singular because function p diverges at $x_0 = -2$ faster than $\dfrac{1}{(x+2)}$. The point $x_1 = 1$ is regular singular because function p is regular at $x_1 = 1$ and function q diverges at $x_1 = 1$ slower than $\dfrac{1}{(x-1)^2}$.

3.3.2. **The Frobenius Method.** We now assume that the differential equation

$$y'' + p(x)\, y' + q(x)\, y = 0, \tag{3.3.1}$$

has a regular singular point. We want to find solutions to this equation that are defined arbitrary close to that regular singular point. Recall that a point x_0 is a regular singular point of the equation above iff the functions $(x - x_0)\, p$ and $(x - x_0)^2\, q$ are analytic at x_0. A function is analytic at a point iff it has a convergent power series expansion in a neighborhood of that point. In our case this means that near a regular singular point holds

$$(x - x_0)\, p(x) = \sum_{n=0}^{\infty} p_n\, (x - x_0)^n = p_0 + p_1(x - x_0) + p_2(x - x_0)^2 + \cdots$$

$$(x - x_0)^2\, q(x) = \sum_{n=0}^{\infty} q_n\, (x - x_0)^n = q_0 + q_1(x - x_0) + q_2(x - x_0)^2 + \cdots$$

This means that near x_0 the function p diverges at most like $(x - x_0)^{-1}$ and function q diverges at most like $(x - x_0)^{-2}$, as it can be seen from the equations

$$p(x) = \frac{p_0}{(x - x_0)} + p_1 + p_2(x - x_0) + \cdots$$

$$q(x) = \frac{q_0}{(x - x_0)^2} + \frac{q_1}{(x - x_0)} + q_2 + \cdots$$

Therefore, for p_0 and q_0 nonzero and x close to x_0 we have the relations

$$p(x) \simeq \frac{p_0}{(x - x_0)}, \qquad q(x) \simeq \frac{q_0}{(x - x_0)^2}, \qquad x \simeq x_0,$$

where the symbol $a \simeq b$, with $a,\, b \in \mathbb{R}$ means that $|a - b|$ is close to zero. In other words, the for x close to a regular singular point x_0 the coefficients of Eq. (3.3.1) are close to the coefficients of the Euler equidimensional equation

$$(x - x_0)^2\, y_e'' + p_0(x - x_0)\, y_e' + q_0\, y_e = 0,$$

where p_0 and q_0 are the zero order terms in the power series expansions of $(x - x_0)\, p$ and $(x - x_0)^2\, q$ given above. One could expect that solutions y to Eq. (3.3.1) are close to solutions y_e to this Euler equation. One way to put this relation in a more precise way is

$$y(x) = y_e(x) \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \Rightarrow \quad y(x) = y_e(x)\, \big(a_0 + a_1(x - x_0) + \cdots\big).$$

Recalling that at least one solution to the Euler equation has the form $y_e(x) = (x - x_0)^r$, where r is a root of the indicial polynomial

$$r(r - 1) + p_0 r + q_0 = 0,$$

we then expect that for x close to x_0 the solution to Eq. (3.3.1) be close to

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

This expression for the solution is usually written in a more compact way as follows,

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{(r+n)}.$$

This is the main idea of the Frobenius method to find solutions to equations with regular singular points. To look for solutions that are close to solutions to an appopriate Euler equation. We now state two theorems summarize a few formulas for solutions to differential equations with regular singular points.

Theorem 3.3.2 (Frobenius). *Assume that the differential equation*

$$y'' + p(x)\,y' + q(x)\,y = 0, \tag{3.3.2}$$

has a regular singular point $x_0 \in \mathbb{R}$ and denote by p_0, q_0 the zero order terms in

$$(x - x_0)\,p(x) = \sum_{n=0}^{\infty} p_n\,(x - x_0)^n, \qquad (x - x_0)^2\,q(x) = \sum_{n=0}^{\infty} q_n\,(x - x_0)^n.$$

Let r_+, r_- be the solutions of the indicial equation

$$r(r-1) + p_0\,r + q_0 = 0.$$

(a) *If $(r_+ - r_-)$ is not an integer, then the differential equation in (3.3.2) has two independent solutions y_+, y_- of the form*

$$y_+(x) = |x - x_0|^{r_+} \sum_{n=0}^{\infty} a_n\,(x - x_0)^n, \quad \text{with} \quad a_0 = 1,$$

$$y_-(x) = |x - x_0|^{r_-} \sum_{n=0}^{\infty} b_n\,(x - x_0)^n, \quad \text{with} \quad b_0 = 1.$$

(b) *If $(r_+ - r_-) = N$, a nonnegative integer, then the differential equation in (3.3.2) has two independent solutions y_+, y_- of the form*

$$y_+(x) = |x - x_0|^{r_+} \sum_{n=0}^{\infty} a_n\,(x - x_0)^n, \quad \text{with} \quad a_0 = 1,$$

$$y_-(x) = |x - x_0|^{r_-} \sum_{n=0}^{\infty} b_n\,(x - x_0)^n + c\,y_+(x)\,\ln|x - x_0|, \quad \text{with} \quad b_0 = 1.$$

The constant c is nonzero if $N = 0$. If $N > 0$, the constant c may or may not be zero.

In both cases above the series converge in the interval defined by $|x - x_0| < \rho$ and the differential equation is satisfied for $0 < |x - x_0| < \rho$.

Remarks:

(a) The statements above are taken from Apostol's second volume [2], Theorems 6.14, 6.15. For a sketch of the proof see Simmons [10]. A proof can be found in [5, 7].

(b) The existence of solutions and their behavior in a neighborhood of a singular point was first shown by Lazarus Fuchs in 1866. The construction of the solution using singular power series expansions was first shown by Ferdinand Frobenius in 1874.

We now give a summary of the Frobenius method to find the solutions mentioned in Theorem 3.3.2 to a differential equation having a regular singular point. For simplicity we only show how to obtain the solution y_+.

(1) Look for a solution y of the form $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$.

(2) Introduce this power series expansion into the differential equation and find the indicial equation for the *exponent* r. Find the larger solution of the indicial equation.

(3) Find a *recurrence relation* for the coefficients a_n.

(4) Introduce the larger root r into the recurrence relation for the coefficients a_n. Only then, solve this latter recurrence relation for the coefficients a_n.

(5) Using this procedure we will find the solution y_+ in Theorem 3.3.2.

We now show how to use these steps to find one solution of a differential equation near a regular singular point. We show the case where the roots of the indicial polynomial differ by an integer. We show that in this case we obtain only solution y_+. The solution y_- does not have the form $y(x) = \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+r)}$. Theorem 3.3.2 says that there is a logarithmic term in the solution. We do not compute that solution here.

Example 3.3.4: Find the solution y near the regular singular point $x_0 = 0$ of the equation

$$x^2\,y'' - x(x+3)\,y' + (x+3)\,y = 0.$$

Solution: We look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n\,x^{(n+r)}.$$

The first and second derivatives are given by

$$y'(x) = \sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r-1)}, \qquad y''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r-2)}.$$

In the case $r = 0$ we had the relation $\sum_{n=0}^{\infty} na_n\,x^{(n-1)} = \sum_{n=1}^{\infty} na_n\,x^{(n-1)}$. But in our case $r \neq 0$, so we do not have the freedom to change in this way the starting value of the summation index n. If we want to change the initial value for n, we have to re-label the summation index. We now introduce these expressions into the differential equation. It is convenient to do this step by step. We start with the term $(x+3)y$, which has the form,

$$\begin{aligned}(x+3)\,y &= (x+3)\sum_{n=0}^{\infty} a_n\,x^{(n+r)}\\ &= \sum_{n=0}^{\infty} a_n\,x^{(n+r+1)} + \sum_{n=0}^{\infty} 3a_n\,x^{(n+r)}\\ &= \sum_{n=1}^{\infty} a_{(n-1)}\,x^{(n+r)} + \sum_{n=0}^{\infty} 3a_n\,x^{(n+r)}. \qquad (3.3.3)\end{aligned}$$

We continue with the term containing y',

$$\begin{aligned}-x(x+3)\,y' &= -(x^2+3x)\sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r-1)}\\ &= -\sum_{n=0}^{\infty}(n+r)a_n\,x^{(n+r+1)} - \sum_{n=0}^{\infty} 3(n+r)a_n\,x^{(n+r)}\\ &= -\sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\,x^{(n+r)} - \sum_{n=0}^{\infty} 3(n+r)a_n\,x^{(n+r)}. \qquad (3.3.4)\end{aligned}$$

Then, we compute the term containing y'' as follows,

$$\begin{aligned}x^2\,y'' &= x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r-2)}\\ &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r)}. \qquad (3.3.5)\end{aligned}$$

As one can see from Eqs.(3.3.3)-(3.3.5), the guiding principle to rewrite each term is to have the power function $x^{(n+r)}$ labeled in the same way on every term. For example, in Eqs.(3.3.3)-(3.3.5) we do not have a sum involving terms with factors $x^{(n+r-1)}$ or factors $x^{(n+r+1)}$. Then, the differential equation can be written as follows,

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n\,x^{(n+r)} - \sum_{n=1}^{\infty}(n+r-1)a_{(n-1)}\,x^{(n+r)}$$
$$-\sum_{n=0}^{\infty}3(n+r)a_n\,x^{(n+r)} + \sum_{n=1}^{\infty}a_{(n-1)}\,x^{(n+r)} + \sum_{n=0}^{\infty}3a_n\,x^{(n+r)} = 0.$$

In the equation above we need to split the sums containing terms with $n \geqslant 0$ into the term $n = 0$ and a sum containing the terms with $n \geqslant 1$, that is,

$$\big[r(r-1) - 3r + 3\big]a_0x^r +$$
$$\sum_{n=1}^{\infty}\Big[(n+r)(n+r-1)a_n - (n+r-1)a_{(n-1)} - 3(n+r)a_n + a_{(n-1)} + 3a_n\Big]\,x^{(n+r)} = 0,$$

and this expression can be rewritten as follows,

$$\big[r(r-1) - 3r + 3\big]a_0x^r +$$
$$\sum_{n=1}^{\infty}\Big[\big[(n+r)(n+r-1) - 3(n+r) + 3\big]a_n - (n+r-1-1)a_{(n-1)}\Big]\,x^{(n+r)} = 0$$

and then,

$$\big[r(r-1) - 3r + 3\big]a_0x^r +$$
$$\sum_{n=1}^{\infty}\Big[\big[(n+r)(n+r-1) - 3(n+r-1)\big]a_n - (n+r-2)a_{(n-1)}\Big]\,x^{(n+r)} = 0$$

hence,

$$\big[r(r-1) - 3r + 3\big]a_0x^r + \sum_{n=1}^{\infty}\Big[(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)}\Big]\,x^{(n+r)} = 0.$$

The *indicial equation* and the *recurrence relation* are given by the equations

$$r(r-1) - 3r + 3 = 0, \tag{3.3.6}$$
$$(n+r-1)(n+r-3)a_n - (n+r-2)a_{(n-1)} = 0. \tag{3.3.7}$$

The way to solve these equations in (3.3.6)-(3.3.7) is the following: First, solve Eq. (3.3.6) for the exponent r, which in this case has two solutions $r_\pm$; second, introduce the first solution r_+ into the recurrence relation in Eq. (3.3.7) and solve for the coefficients a_n; the result is a solution y_+ of the original differential equation; then introduce the second solution r_- into Eq. (3.3.7) and solve again for the coefficients a_n; the new result is a second solution y_-. Let us follow this procedure in the case of the equations above:

$$r^2 - 4r + 3 = 0 \quad\Rightarrow\quad r_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-12}\big] \quad\Rightarrow\quad \begin{cases} r_+ = 3, \\ r_- = 1. \end{cases}$$

Introducing the value $r_+ = 3$ into Eq. (3.3.7) we obtain

$$(n+2)n\,a_n - (n+1)a_{n-1} = 0.$$

One can check that the solution y_+ obtained form this recurrence relation is given by

$$y_+(x) = a_0\,x^3\Big[1 + \frac{2}{3}\,x + \frac{1}{4}\,x^2 + \frac{1}{15}\,x^3 + \cdots\Big].$$

Notice that $r_+ - r_- = 3 - 1 = 2$, this is a nonpositive integer. Theorem 3.3.2 says that the solution y_- contains a logarithmic term. Therefore, the solution y_- is not of the form $\sum_{n=0}^{\infty} a_n x^{(r+n)}$, as we have assumed in the calculations done in this example. But, what does happen if we continue this calculation for $r_- = 1$? What solution do we get? Let us find out. We introduce the value $r_- = 1$ into Eq. (3.3.7), then we get

$$n(n-2)a_n - (n-1)a_{n-1} = 0.$$

One can also check that the solution $\tilde{y}_-$ obtained form this recurrence relation is given by

$$\begin{aligned}\tilde{y}_-(x) &= a_2\, x\Big[x^2 + \frac{2}{3}\,x^3 + \frac{1}{4}\,x^4 + \frac{1}{15}\,x^5 + \cdots\Big],\\ &= a_2\, x^3\Big[1 + \frac{2}{3}\,x + \frac{1}{4}\,x^2 + \frac{1}{15}\,x^3 + \cdots\Big] \quad\Rightarrow\quad \tilde{y}_- = \frac{a_2}{a_1}\,y_+.\end{aligned}$$

So get a solution, but this solution is proportional to y_+. To get a solution not proportional to y_+ we need to add the logarithmic term, as in Theorem 3.3.2. ◁

3.3.3. **The Bessel Equation.** We saw in § 3.1 that the Legendre equation appears when one solves the Laplace equation in spherical coordinates. If one uses cylindrical coordinates insted, one needs to solve the Bessel equation. Recall we mentioned that the Laplace equation describes several phenomena, such as the static electric potential near a charged body, or the gravitational potential of a planet or star. When the Laplace equation describes a situation having cylindrical symmetry it makes sense to use cylindrical coordinates to solve it. Then the Bessel equation appears for the radial variable in the cylindrical coordinate system. See Jackson's classic book on electrodynamics [8], § 3.7, for a derivation of the Bessel equation from the Laplace equation.

The equation is named after Friedrich Bessel, a German astronomer from the first half of the seventeen century, who was the first person to calculate the distance to a star other than our Sun. Bessel's parallax of 1838 yielded a distance of 11 light years for the star 61 Cygni. In 1844 he discovered that Sirius, the brightest star in the sky, has a traveling companion. Nowadays such system is called a binary star. This companion has the size of a planet and the mass of a star, so it has a very high density, many thousand times the density of water. This was the first dead start discovered. Bessel first obtained the equation that now bears his name when he was studing star motions. But the equation first appeared in Daniel Bernoulli's studies of oscillations of a hanging chain. (Taken from Simmons' book [10], § 34.)

Example 3.3.5: Find all solutions $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$, with $a_0 \neq 0$, of the Bessel equation

$$x^2\, y'' + x\, y' + (x^2 - \alpha^2)\, y = 0, \qquad x > 0,$$

where α is any real nonnegative constant, using the Frobenius method centered at $x_0 = 0$.

Solution: Let us double check that $x_0 = 0$ is a regular singular point of the equation. We start writing the equation in the standard form,

$$y'' + \frac{1}{x}\, y' + \frac{(x^2 - \alpha^2)}{x^2}\, y = 0,$$

so we get the functions $p(x) = 1/x$ and $q(x) = (x^2 - \alpha^2)/x^2$. It is clear that $x_0 = 0$ is a singular point of the equation. Since the functions

$$\tilde{p}(x) = xp(x) = 1, \qquad \tilde{q}(x) = x^2 q(x) = (x^2 - \alpha^2)$$

are analytic, we conclude that $x_0 = 0$ is a regular singular point. So it makes sense to look for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r)}, \qquad x > 0.$$

We now compute the different terms needed to write the differential equation. We need,

$$x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{(n+r+2)} \quad \Rightarrow \quad y(x) = \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)},$$

where we did the relabeling $n + 2 = m \to n$. The term with the first derivative is given by

$$x\, y'(x) = \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r)}.$$

The term with the second derivative has the form

$$x^2\, y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)}.$$

Therefore, the differential equation takes the form

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{(n+r)} + \sum_{n=0}^{\infty} (n+r) a_n x^{(n+r)}$$
$$+ \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} - \sum_{n=0}^{\infty} \alpha^2\, a_n x^{(n+r)} = 0.$$

Group together the sums that start at $n = 0$,

$$\sum_{n=0}^{\infty} \big[(n+r)(n+r-1) + (n+r) - \alpha^2\big] a_n\, x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)},$$

and cancel a few terms in the first sum,

$$\sum_{n=0}^{\infty} \big[(n+r)^2 - \alpha^2\big] a_n\, x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)} x^{(n+r)} = 0.$$

Split the sum that starts at $n = 0$ into its first two terms plus the rest,

$$(r^2 - \alpha^2) a_0\, x^r + \big[(r+1)^2 - \alpha^2\big] a_1\, x^{(r+1)}$$
$$+ \sum_{n=2}^{\infty} \big[(n+r)^2 - \alpha^2\big] a_n\, x^{(n+r)} + \sum_{n=2}^{\infty} a_{(n-2)}\, x^{(n+r)} = 0.$$

The reason for this splitting is that now we can write the two sums as one,

$$(r^2 - \alpha^2) a_0\, x^r + \big[(r+1)^2 - \alpha^2\big] a_1\, x^{(r+1)} + \sum_{n=2}^{\infty} \Big\{ \big[(n+r)^2 - \alpha^2\big] a_n + a_{(n-2)} \Big\}\, x^{(n+r)} = 0.$$

We then conclude that each term must vanish,

$$(r^2 - \alpha^2) a_0 = 0, \quad \big[(r+1)^2 - \alpha^2\big] a_1 = 0, \quad \big[(n+r)^2 - \alpha^2\big] a_n + a_{(n-2)} = 0, \quad n \geqslant 2. \tag{3.3.8}$$

This is the recurrence relation for the Bessel equation. It is here where we use that we look for solutions with $a_0 \neq 0$. In this example we do not look for solutions with $a_1 \neq 0$. Maybe it is a good exercise for the reader to find such solutions. But in this example we look for solutions with $a_0 \neq 0$. This condition and the first equation above imply that

$$r^2 - \alpha^2 = 0 \quad \Rightarrow \quad r_\pm = \pm\alpha,$$

and recall that α is a nonnegative but otherwise arbitrary real number. The choice $r = r_+$ will lead to a solution y_α, and the choice $r = r_-$ will lead to a solution $y_{-\alpha}$. These solutions may or may not be linearly independent. This depends on the value of α, since $r_+ - r_- = 2\alpha$. One must be careful to study all possible cases.

Remark: Let us start with a very particular case. Suppose that both equations below hold,

$$(r^2 - \alpha^2) = 0, \qquad \big[(r+1)^2 - \alpha^2\big] = 0.$$

This equations are the result of both $a_0 \neq 0$ and $a_1 \neq 0$. These equations imply

$$r^2 = (r+1)^2 \quad \Rightarrow \quad 2r + 1 = 0 \quad \Rightarrow \quad r = -\frac{1}{2}.$$

But recall that $r = \pm\alpha$, and $\alpha \geqslant 0$, hence the case $a_0 \neq 0$ and $a_1 \neq 0$ happens only when $\alpha = 1/2$ and we choose $r_- = -\alpha = -1/2$. We leave computation of the solution $y_{-1/2}$ as an exercise for the reader. But the answer is

$$y_{-1/2}(x) = a_0\,\frac{\cos(x)}{\sqrt{x}} + a_1\,\frac{\sin(x)}{\sqrt{x}}.$$

From now on we assume that $\alpha \neq 1/2$. This condition on α, the equation $r^2 - \alpha^2 = 0$, and the remark above imply that

$$(r+1)^2 - \alpha^2 \neq 0.$$

So the second equation in the recurrence relation in (3.3.8) implies that $a_1 = 0$. Summarizing, the first two equations in the recurrence relation in (3.3.8) are satisfied because

$$r_\pm = \pm\alpha, \qquad a_1 = 0.$$

We only need to find the coefficients a_n, for $n \geqslant 2$ such that the third equation in the recurrence relation in (3.3.8) is satisfied. But we need to consider two cases, $r = r_+ = \alpha$ and $r_- = -\alpha$.

We start with the case $r = r_+ = \alpha$, and we get

$$(n^2 + 2n\alpha)\,a_n + a_{(n-2)} = 0 \quad \Rightarrow \quad n(n+2\alpha)\,a_n = -a_{(n-2)}.$$

Since $n \geqslant 2$ and $\alpha \geqslant 0$, the factor $(n+2\alpha)$ never vanishes and we get

$$a_n = -\frac{a_{(n-2)}}{n(n+2\alpha)}.$$

This equation and $a_1 = 0$ imply that all coefficients $a_{2k+1} = 0$ for $k \geqslant 0$, the odd coefficients vanish. On the other hand, the even coefficent are nonzero. The coefficient a_2 is

$$a_2 = -\frac{a_0}{2(2+2\alpha)} \quad \Rightarrow \quad a_2 = -\frac{a_0}{2^2(1+\alpha)},$$

the coefficient a_4 is

$$a_4 = -\frac{a_2}{4(4+2\alpha)} = -\frac{a_2}{2^2(2)(2+\alpha)} \quad \Rightarrow \quad a_4 = \frac{a_0}{2^4(2)(1+\alpha)(2+\alpha)},$$

the coefficient a_6 is

$$a_6 = -\frac{a_4}{6(6+2\alpha)} = -\frac{a_4}{2^2(3)(3+\alpha)} \quad \Rightarrow \quad a_6 = -\frac{a_0}{2^6(3!)(1+\alpha)(2+\alpha)(3+\alpha)}.$$

Now it is not so hard to show that the general term a_{2k}, for $k = 0, 1, 2, \cdots$ has the form

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k}(k!)(1+\alpha)(2+\alpha)\cdots(k+\alpha)}.$$

We then get the solution y_α

$$y_\alpha(x) = a_0\, x^\alpha \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} (k!) (1+\alpha)(2+\alpha)\cdots(k+\alpha)}\Big], \qquad \alpha \geqslant 0. \tag{3.3.9}$$

The ratio test shows that this power series converges for all $x \geqslant 0$. When $a_0 = 1$ the corresponding solution is usually called in the literature as J_α,

$$J_\alpha(x) = x^\alpha \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} (k!) (1+\alpha)(2+\alpha)\cdots(k+\alpha)}\Big], \qquad \alpha \geqslant 0.$$

We now look for solutions to the Bessel equation coming from the choice $r = r_- = -\alpha$, with $a_1 = 0$, and $\alpha \neq 1/2$. The third equation in the recurrence relation in (3.3.8) implies

$$(n^2 - 2n\alpha) a_n + a_{(n-2)} = 0 \quad \Rightarrow \quad n(n-2\alpha) a_n = -a_{(n-2)}.$$

If $2\alpha = N$, a nonnegative integer, the second equation above implies that the recurrence relation cannot be solved for a_n with $n \geqslant N$. This case will be studied later on. Now assume that 2α is not a nonnegative integer. In this case the factor $(n - 2\alpha)$ never vanishes and

$$a_n = -\frac{a_{(n-2)}}{n(n-2\alpha)}.$$

This equation and $a_1 = 0$ imply that all coefficients $a_{2k+1} = 0$ for $k \geqslant 0$, the odd coefficients vanish. On the other hand, the even coefficent are nonzero. The coefficient a_2 is

$$a_2 = -\frac{a_0}{2(2-2\alpha)} \quad \Rightarrow \quad a_2 = -\frac{a_0}{2^2(1-\alpha)},$$

the coefficient a_4 is

$$a_4 = -\frac{a_2}{4(4-2\alpha)} = -\frac{a_2}{2^2(2)(2-\alpha)} \quad \Rightarrow \quad a_4 = \frac{a_0}{2^4(2)(1-\alpha)(2-\alpha)},$$

the coefficient a_6 is

$$a_6 = -\frac{a_4}{6(6-2\alpha)} = -\frac{a_4}{2^2(3)(3-\alpha)} \quad \Rightarrow \quad a_6 = -\frac{a_0}{2^6(3!)(1-\alpha)(2-\alpha)(3-\alpha)}.$$

Now it is not so hard to show that the general term a_{2k}, for $k = 0, 1, 2, \cdots$ has the form

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} (k!) (1-\alpha)(2-\alpha)\cdots(k-\alpha)}.$$

We then get the solution $y_{-\alpha}$

$$y_{-\alpha}(x) = a_0\, x^\alpha \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} (k!) (1-\alpha)(2-\alpha)\cdots(k-\alpha)}\Big], \qquad \alpha \geqslant 0. \tag{3.3.10}$$

The ratio test shows that this power series converges for all $x \geqslant 0$. When $a_0 = 1$ the corresponding solution is usually called in the literature as $J_{-\alpha}$,

$$J_{-\alpha}(x) = x^{-\alpha} \Big[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} (k!) (1-\alpha)(2-\alpha)\cdots(k-\alpha)}\Big], \qquad \alpha \geqslant 0.$$

The function $y_{-\alpha}$ was obtained assuming that 2α is not a nonnegative integer. From the calculations above it is clear that we need this condition on α so we can compute a_n in terms of $a_{(n-2)}$. Notice that $r_\pm = \pm\alpha$, hence $(r_+ - r_-) = 2\alpha$. So the condition on α is the condition $(r_+ - r_-)$ not a nonnegative integer, which appears in Theorem 3.3.2.

However, there is something special about the Bessel equation. That the constant 2α is not a nonnegative integer means that α is neither an integer nor an integer plus one-half. But the formula for $y_{-\alpha}$ is well defined even when α is an integer plus one-half, say $k + 1/2$, for

k integer. Introducing this $y_{-(k+1/2)}$ into the Bessel equation one can check that $y_{-(k+1/2)}$ is a solution to the Bessel equation.

Summarizing, the solutions of the Bessel equation function y_α is defined for every nonnegative real number α, and $y_{-\alpha}$ is defined for every nonnegative real number α except for nonnegative integers. For a given α such that both y_α and $y_{-\alpha}$ are defined, these functions are linearly independent. That these functions cannot be proportional to each other is simple to see, since for $\alpha > 0$ the function y_α is regular at the origin $x = 0$, while $y_{-\alpha}$ diverges.

The last case we need to study is how to find the solution $y_{-\alpha}$ when α is a nonnegative integer. We see that the expression in (3.3.10) is not defined when α is a nonnegative integer. And we just saw that this condition on α is a particular case of the condition in Theorem 3.3.2 that $(r_+ - r_-)$ is not a nonnegative integer. Theorem 3.3.2 gives us what is the expression for a second solution, $y_{-\alpha}$ linearly independent of y_α, in the case that α is a nonnegative integer. This expression is

$$y_{-\alpha}(x) = y_\alpha(x)\, \ln(x) + x^{-\alpha} \sum_{n=0}^{\infty} c_n x^n.$$

If we put this expression into the Bessel equation, one can find a recurrence relation for the coefficients c_n. This is a long calculation, and the final result is

$$\begin{aligned} y_{-\alpha}(x) &= y_\alpha(x)\, \ln(x) \\ &\quad - \frac{1}{2}\Big(\frac{x}{2}\Big)^{-\alpha} \sum_{n=0}^{\alpha-1} \frac{(\alpha-n-1)!}{n!} \Big(\frac{x}{2}\Big)^{2n} \\ &\quad - \frac{1}{2}\Big(\frac{x}{2}\Big)^{\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{(h_n + h_{(n+\alpha)})}{n!\,(n+\alpha)!} \Big(\frac{x}{2}\Big)^{2n}, \end{aligned}$$

with $h_0 = 0$, $h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ for $n \geqslant 1$, and α a nonnegative integer. ◁

3.3.4. Exercises.

3.3.1.- .

3.3.2.- .

Notes on Chapter 3

Sometimes solutions to a differential equation cannot be written in terms of previously known functions. When that happens the we say that the solutions to the differential equation define a new type of functions. How can we work with, or let alone write down, a new function, a function that cannot be written in terms of the functions we already know? It is the differential equation what defines the function. So the function properties must be obtained from the differential equation itself. A way to compute the function values must come from the differential equation as well. The few paragraphs that follow try to give sense that this procedure is not as artificial as it may sound.

Differential Equations to Define Functions. We have seen in § 3.3 that the solutions of the Bessel equation for $\alpha \neq 1/2$ cannot be written in terms of simple functions, such as quotients of polynomials, trigonometric functions, logarithms and exponentials. We used power series including negative powers to write solutions to this equation. To study properties of these solutions one needs to use either the power series expansions or the equation itself. This type of study on the solutions of the Bessel equation is too complicated for these notes, but the interested reader can see [14].

We want to give an idea how this type of study can be carried out. We choose a differential equation that is simpler to study than the Bessel equation. We study two solutions, C and S, of this particular differential equation and we will show, using only the differential equation, that these solutions have all the properties that the cosine and sine functions have. So we will conclude that these solutions are in fact $C(x) = \cos(x)$ and $S(x) = \sin(x)$. This example is taken from Hassani's textbook [6], example 13.6.1, page 368.

Example 3.3.6: Let the function C be the unique solution of the initial value problem

$$C'' + C = 0, \qquad C(0) = 1, \quad C'(0) = 0,$$

and let the function S be the unique solution of the initial value problem

$$S'' + S = 0, \qquad S(0) = 0, \quad S'(0) = 1.$$

Use the differential equation to study these functions.

Solution:

(a) We start showing that these solutions C and S are linearly independent. We only need to compute their Wronskian at $x = 0$.

$$W(0) = C(0)\, S'(0) - C'(0)\, S(0) = 1 \neq 0.$$

Therefore the functions C and S are linearly independent.

(b) We now show that the function S is odd and the function C is even. The function $\hat{C}(x) = C(-x)$ satisfies the initial value problem

$$\hat{C}'' + \hat{C} = C'' + C = 0, \qquad \hat{C}(0) = C(0) = 1, \quad \hat{C}'(0) = -C'(0) = 0.$$

This is the same initial value problem satisfied by the function C. The uniqueness of solutions to these initial value problem implies that $C(-x) = C(x)$ for all $x \in \mathbb{R}$, hence the function C is even. The function $\hat{S}(x) = S(-x)$ satisfies the initial value problem

$$\hat{S}'' + \hat{S} = S'' + S = 0, \qquad \hat{S}(0) = S(0) = 0, \quad \hat{S}'(0) = -S'(0) = -1.$$

This is the same initial value problem satisfied by the function $-S$. The uniqueness of solutions to these initial value problem implies that $S(-x) = -S(x)$ for all $x \in \mathbb{R}$, hence the function S is odd.

(c) Next we find a differential relation between the functions C and S. Notice that the function $-C'$ is the uique solution of the initial value problem

$$(-C')'' + (-C') = 0, \qquad -C'(0) = 0, \quad (-C')'(0) = C(0) = 1.$$

This is precisely the same initial value problem satisfied by the function S. The uniqueness of solutions to these initial value problems implies that $-C = S$, that is for all $x \in \mathbb{R}$ holds

$$C'(x) = -S(x).$$

Take one more derivative in this relation and use the differential equation for C,

$$S'(x) = -C''(x) = C(x) \quad \Rightarrow \quad S'(x) = C(x).$$

(d) Let us now recall that Abel's Theorem says that the Wronskian of two solutions to a second order differential equation $y'' + p(x)\,y' + q(x)\,y = 0$ satisfies the differential equation $W' + p(x)\,W = 0$. In our case the function $p = 0$, so the Wronskian is a constant function. If we compute the Wronskian of the functions C and S and we use the differential relations found in (c) we get

$$W(x) = C(x)\,S'(x) - C'(x)\,S(x) = C^2(x) + S^2(x).$$

This Wronskian must be a constant function, but at $x = 0$ takes the value $W(0) = C^2(0) + S^2(0) = 1$. We therefore conclude that for all $x \in \mathbb{R}$ holds

$$C^2(x) + S^2(x) = 1.$$

(e) We end computing power series expansions of these functions C and S, so we have a way to compute their values. We start with function C. The initial conditions say

$$C(0) = 1, \quad C'(0) = 0.$$

The differential equation at $x = 0$ and the first initial condition say that $C''(0) = -C(0) = -1$. The derivative of the differential equation at $x = 0$ and the second initial condition say that $C'''(0) = -C'(0) = 0$. If we keep taking derivatives of the differential equation we get

$$C''(0) = -1, \quad C'''(0) = 0, \quad C^{(4)}(0) = 1,$$

and in general,

$$C^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^k & \text{if } n = 2k, \text{ where } k = 0, 1, 2, \cdots. \end{cases}$$

So we obtain the Taylor series expansion

$$C(x) = \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k}}{(2k)!},$$

which is the power series expansion of the cosine function. A similar calculation yields

$$S(x) = \sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!},$$

which is the power series expansion of the sine function. Notice that we have obtained these expansions using only the differential equation and its derivatives at $x = 0$ together with the initial conditions. The ratio test shows that these power series converge for all $x \in \mathbb{R}$. These power series expansions also say that the function S is odd and C is even. ◁

Review of Natural Logarithms and Exponentials. The discovery, or invention, of a new type of functions happened many times before the time of differential equations. Looking at the history of mathematics we see that people first defined polynomials as additions and multiplications on the independent variable x. After that came quotient of polynomials. Then people defined trigonometric functions as ratios of geometric objects. For example the sine and cosine functions were originally defined as ratios of the sides of right triangles. These were all the functions known before calculus, before the seventeen century. Calculus brought the natural logarithm and its inverse, the exponential function together with the number e.

What is used to define the natural logarithm is not a differential equation but integration. People knew that the antiderivative of a power function $f(x) = x^n$ is another power function $F(x) = x^{(n+1)}/(n+1)$, except for $n = -1$, where this rule fails. The antiderivative of the function $f(x) = 1/x$ is neither a power function nor a trigonometric function, so at that time it was a new function. People gave a name to this new function, ln, and defined it as whatever comes from the integration of the function $f(x) = 1/x$, that is,

$$\ln(x) = \int_1^x \frac{ds}{s}, \qquad x > 0.$$

All the properties of this new function must come from that definition. It is clear that this function is increasing, that $\ln(1) = 0$, and that the function take values in $(-\infty, \infty)$. But this function has a more profound property, $\ln(ab) = \ln(a) + \ln(b)$. To see this relation first compute

$$\ln(ab) = \int_1^{ab} \frac{ds}{s} = \int_1^a \frac{ds}{s} + \int_a^{ab} \frac{ds}{s};$$

then change the variable in the second term, $\tilde{s} = s/a$, so $d\tilde{s} = ds/a$, hence $ds/s = d\tilde{s}/\tilde{s}$, and

$$\ln(ab) = \int_1^a \frac{ds}{s} + \int_1^b \frac{d\tilde{s}}{\tilde{s}} = \ln(a) + \ln(b).$$

The Euler number e is defined as the solution of the equation $\ln(e) = 1$. The inverse of the natural logarithm, $\ln^{-1}$, is defined in the usual way,

$$\ln^{-1}(y) = x \quad \Leftrightarrow \quad \ln(x) = y, \qquad x \in (0, \infty), \quad y \in (-\infty, \infty).$$

Since the natural logarithm satisfies that $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$, the inverse function satisfies the related identity $\ln^{-1}(y_1 + y_2) = \ln^{-1}(y_1)\, \ln^{-1}(y_2)$. To see this identity compute

$$\ln^{-1}(y_1 + y_2) = \ln^{-1}\big(\ln(x_1) + \ln(x_2)\big) = \ln^{-1}(\ln(x_1 x_2)) = x_1 x_2 = \ln^{-1}(y_1)\, \ln^{-1}(y_2).$$

This identity and the fact that $\ln^{-1}(1) = e$ imply that for any positive integer n holds

$$\ln^{-1}(n) = \ln^{-1}\,(\overbrace{1 + \cdots + 1}^{\text{n times}}) = \overbrace{\ln^{-1}(1) \cdots \ln^{-1}(1)}^{\text{n times}} = \overbrace{e \cdots e}^{\text{n times}} = e^n.$$

This relation says that $\ln^{-1}$ is the exponential function when restricted to positive integers. This suggests a way to generalize the exponential function from positive integers to real numbers, $e^y = \ln^{-1}(y)$, for y real. Hence the name exponential for the inverse of the natural logarithm. And this is how calculus brought us the logarithm and the exponential functions.

Finally notice that by the definition of the natural logarithm, its derivative is $\ln'(x) = 1/x$. But there is a formula relating the derivative of a function f and its inverse f^{-1},

$$\big(f^{-1}\big)'(y) = \frac{1}{f'\big(f^{-1}(y)\big)}.$$

Using this formula for the natural logarithm we see that

$$\big(\ln^{-1}\big)'(y) = \frac{1}{\ln'\big(\ln^{-1}(y)\big)} = \ln^{-1}(y).$$

In other words, the inverse of the natural logarithm, call it now $g(y) = \ln^{-1}(y) = e^y$, must be a solution to the differential equation

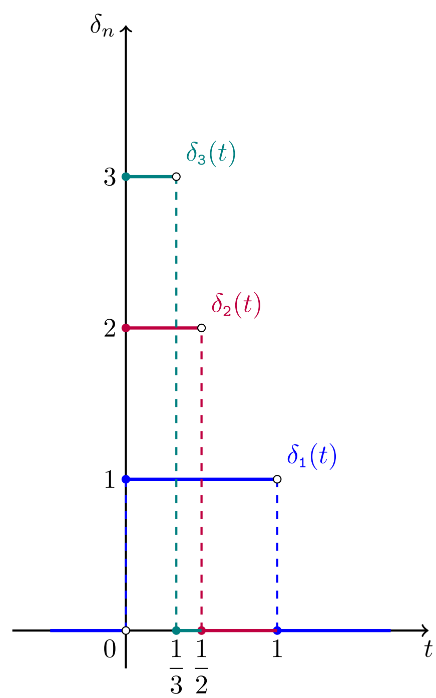
$$g'(y) = g(y).$$

And this is how logarithms and exponentials can be added to the set of known functions. Of course, now that we know about differential equations, we can always start with the differential equation above and obtain all the properties of the exponential function using the differential equation. This might be a nice exercise for the interested reader.

Chapter 4. The Laplace Transform Method

The Laplace Transform is a transformation, meaning that it changes a function into a new function. Actually, it is a linear transformation, because it converts a linear combination of functions into a linear combination of the transformed functions. Even more interesting, the Laplace Transform converts derivatives into multiplications. These two properties make the Laplace Transform very useful to solve linear differential equations with constant coefficients. The Laplace Transform converts such differential equation for an unknown function into an algebraic equation for the transformed function. Usually it is easy to solve the algebraic equation for the transformed function. Then one converts the transformed function back into the original function. This function is the solution of the differential equation.

Solving a differential equation using a Laplace Transform is radically different from all the methods we have used so far. This method, as we will use it here, is relatively new. The Laplace Transform we define here was first used in 1910, but its use grew rapidly after 1920, specially to solve differential equations. Transformations like the Laplace Transform were known much earlier. Pierre Simon de Laplace used a similar transformation in his studies of probability theory, published in 1812, but analogous transformations were used even earlier by Euler around 1737.



4.1. Definition of the Laplace Transform

The Laplace Transform is an integral transform. It is defined by an improper integral. So we start this Section with a brief review on improper integrals. Then we define the Laplace Transform. In the following sections we explain how to use this transform to find solutions to differential equations. The Laplace Transform is specially useful to solve linear non-homogeneous differential equations with constant coefficients.

4.1.1. **Review of Improper Integrals.**

Improper integrals are integrals on unbounded domains. They are defined as a limit of definite integrals. More precisely,

$$\int_{t_0}^{\infty} g(t)\,dt = \lim_{N\to\infty} \int_{t_0}^{N} g(t)\,dt.$$

We say that the integral above *converges* iff the limit exists, otherwise we say that the integral *diverges*. In the following example we compute an improper integral that is very useful to compute Laplace Transforms.

Example 4.1.1: Compute the improper integral $I = \displaystyle\int_0^{\infty} e^{-at}\,dt$, with $a \in \mathbb{R}$.

Solution: Following the definition above we need to first compute a definite integral and then take a limit. So, from the definition,

$$I = \int_0^{\infty} e^{-at}\,dt = \lim_{N\to\infty} \int_0^{N} e^{-at}\,dt.$$

We first compute the definite integral. We start with the case $a = 0$,

$$I = \lim_{N\to\infty} \int_0^{N} dt = \lim_{N\to\infty} t\Big|_0^N = \lim_{N\to\infty} N = \infty,$$

therefore for $a = 0$ the improper integral I does not exist. When $a \neq 0$ we have

$$I = \lim_{N\to\infty} -\frac{1}{a}\Big(e^{-aN} - 1\Big).$$

In the case $a < 0$, that is $a = -|a|$, we have that

$$\lim_{N\to\infty} e^{|a|N} = \infty \quad \Rightarrow \quad I = -\infty,$$

therefore for $a < 0$ the improper integral I does not exist. In the case $a > 0$ we know that

$$\lim_{N\to\infty} e^{-aN} = 0 \quad \Rightarrow \quad \int_0^{\infty} e^{-at}\,dt = \frac{1}{a}, \quad a > 0. \tag{4.1.1}$$

◁

4.1.2. **Definition and Table.**

The Laplace Transform is a transformation, meaning that it converts a function into a new function. We have seen transformations earlier in these notes. In Chapter 2 we used the transformation

$$L[y(t)] = y''(t) + a_1\,y'(t) + a_0\,y(t),$$

so that a second order linear differential equation with source f could be written as $L[y] = f$. There are simpler transformations, for example the differentiation operation itself,

$$D[f(t)] = f'(t).$$

Not all transformations involve differentiation. There are integral transformations, for example integration itself,

$$I[f(t)] = \int_0^x f(t)\,dt.$$

Of particular importance in many applications are integral transformations of the form

$$T[f(t)] = \int_a^b K(s,t)\, f(t)\, dt,$$

where K is a fixed function of two variables, called the *kernel* of the transformation, and a, b are real numbers or $\pm\infty$. The Laplace Transform is a transfomation of this type, where the kernel is $K(s,t) = e^{-st}$, the constant $a = 0$, and $b = \infty$.

Definition 4.1.1. *The* ***Laplace Transform*** *of a function* $f : (0,\infty) \to \mathbb{R}$ *is given by*

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt, \tag{4.1.2}$$

where $s \in \mathbb{R}$ *is any real number such that the integral above converges.*

Remark: An alternative notation for the Laplace Transform of a function f is

$$F(s) = \mathcal{L}[f(t)], \qquad s \in D_F \subset \mathbb{R},$$

where the emphasis is in the result of the Laplace Transform, which is a function F on the variable s. We have denoted the domain of the transformed function as $D_F \subset \mathbb{R}$, defined as the set of all real numbers such that the integral in (4.1.2) converges. In these notes we use both notations $\mathcal{L}[f(t)]$ and F, depending on what we want to emphasize, either the transformation itself or the result of the transformation. We will also use the notation $\mathcal{L}[f]$, whenever the independent variable t is not important in that context.

In this Section we study properties of the transformation $\mathcal{L}$. We will show in Theorem 4.1.4 that this transformation is linear, and in Theorem 4.2.1 that this transformation is one-to-one and so invertible on the appropriate domain. But before that, we show how to compute a Laplace Transform, how to compute the improper integral and interpret the result. We will see in a few examples below that this improper integral in Eq. (4.1.2) does not converge for every $s \in \mathbb{R}$. The interval where the Laplace Transform of a function f is defined depends on the particular function f. We will see that $\mathcal{L}[e^{at}]$ with $a \in \mathbb{R}$ is defined for $s > a$, but $\mathcal{L}[\sin(at)]$ is defined for $s > 0$. Let us compute a few Laplace Transforms.

EXAMPLE 4.1.2: Compute $\mathcal{L}[1]$.

SOLUTION: The function $f(t) = 1$ is a simple enough function to find its Laplace transform. Following the definition,

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt.$$

But we have computed this improper integral in Example 4.1.1. Just replace $a = s$ in that example. The result is that the $\mathcal{L}[1]$ is not defined for $s \leqslant 0$, while for $s > 0$ we have

$$\mathcal{L}[1] = \frac{1}{s}, \qquad s > 0.$$

$\lhd$

EXAMPLE 4.1.3: Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

SOLUTION: Following the definition of the Laplace Transform,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt.$$

Here again we can use the result in Example 4.1.1, just replace a in that Example by $(s-a)$. The result is that the $\mathcal{L}[e^{at}]$ is not defined for $s \leqslant a$, while for $s > a$ we have

$$\mathcal{L}[e^{at}] = \frac{1}{(s-a)}, \qquad s > a.$$

$\lhd$

Example 4.1.4: Compute $\mathcal{L}[te^{at}]$, where $a \in \mathbb{R}$.

Solution: In this case the calculation is more complicated than above, since we need to integrate by parts. We start with the definition of the Laplace Transform,

$$\mathcal{L}[te^{at}] = \int_0^\infty e^{-st} te^{at}\, dt = \lim_{N\to\infty} \int_0^N te^{-(s-a)t}\, dt.$$

This improper integral diverges for $s = a$, so $\mathcal{L}[te^{at}]$ is not defined for $s = a$. From now on we consider only the case $s \neq a$. In this case we can integrate by parts,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N + \frac{1}{s-a} \int_0^N e^{-(s-a)t}\, dt \Big],$$

that is,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N - \frac{1}{(s-a)^2} e^{-(s-a)t}\Big|_0^N \Big]. \qquad (4.1.3)$$

In the case that $s < a$ the first term above diverges,

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{-(s-a)N} = \lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{|s-a|N} = \infty,$$

therefore $\mathcal{L}[te^{at}]$ is not defined for $s < a$. In the case $s > a$ the first term on the right hand side in (4.1.3) vanishes, since

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)} t\, e^{-(s-a)t}\big|_{t=0} = 0.$$

Regarding the other term, and recalling that $s > a$,

$$\lim_{N\to\infty} -\frac{1}{(s-a)^2} e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)^2} e^{-(s-a)t}\big|_{t=0} = \frac{1}{(s-a)^2}.$$

Therefore, we conclude that

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}, \qquad s > a.$$

◁

Example 4.1.5: Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

Solution: In this case we need to compute

$$\mathcal{L}[\sin(at)] = \lim_{N\to\infty} \int_0^N e^{-st} \sin(at)\, dt.$$

The definite integral above can be computed integrating by parts twice,

$$\int_0^N e^{-st} \sin(at)\, dt = -\frac{1}{s} \big[e^{-st} \sin(at) \big]\Big|_0^N - \frac{a}{s^2} \big[e^{-st} \cos(at) \big]\Big|_0^N - \frac{a}{s^2} \int_0^N e^{-st} \sin(at)\, dt,$$

which implies that

$$\Big(1 + \frac{a}{s^2}\Big) \int_0^N e^{-st} \sin(at)\, dt = -\frac{1}{s} \big[e^{-st} \sin(at) \big]\Big|_0^N - \frac{a}{s^2} \big[e^{-st} \cos(at) \big]\Big|_0^N.$$

One can check that the limit $N \to \infty$ on the right hand side above does not exist for $s \leqslant 0$, so $\mathcal{L}[\sin(at)]$ does not exist for $s \leqslant 0$. In the case $s > 0$ it is not difficult to see that

$$\Big(\frac{s^2 + a^2}{s^2}\Big) \int_0^\infty e^{-st} \sin(at)\, dt = \frac{a}{s^2},$$

which is equivalent to

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \qquad s > 0.$$

◁

In Table 2 we present a short list of Laplace Transforms. They can be computed in the same way we computed the the Laplace Transforms in the examples above.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	D_F
$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > \|a\|$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > \|a\|$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2 - b^2}$	$s - a > \|b\|$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \|b\|$

TABLE 2. List of a few Laplace Transforms.

4.1.3. **Main Properties.** Since we are more or less confident on how to compute a Laplace Transform, we can start asking deeper questions. For example, what type of functions have a Laplace Transform? It turns out that a large class of functions, those that are piecewise continuous on $[0, \infty)$ and bounded by an exponential. This last property is particularly important and we give it a name.

Definition 4.1.2. *A function f on $[0, \infty)$ is of* ***exponential order*** *s_0, where s_0 is any real number, iff there exist positive constants k, T such that*

$$|f(t)| \leqslant k\, e^{s_0 t} \quad \textit{for all} \quad t > T. \tag{4.1.4}$$

Remarks:

(a) When the precise value of the constant s_0 is not important we will say that f is of exponential order.

(b) An example of a function that is not of exponential order is $f(t) = e^{t^2}$.

This definition helps to describe a set of functions having Laplace Transform. Piecewise continuous functions on $[0, \infty)$ of exponential order have Laplace Transforms.

Theorem 4.1.3 (Sufficient Conditions). *If the function f on $[0, \infty)$ is piecewise continuous and of exponential order s_0, then the $\mathcal{L}[f]$ exists for all $s > s_0$ and there exists a positive constant k such that the following bound holds*

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s - s_0}, \qquad s > s_0.$$

Proof of Theorem 4.1.3: From the definition of the Laplace Transform we know that

$$\mathcal{L}[f] = \lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt.$$

The definite integral on the interval $[0, N]$ exists for every $N > 0$ since f is piecewise continuous on that interval, no matter how large N is. We only need to check whether the integral converges as $N \to \infty$. This is the case for functions of exponential order, because

$$\Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant \int_0^N e^{-st} |f(t)|\, dt \leqslant \int_0^N e^{-st} k e^{s_0 t}\, dt = k \int_0^N e^{-(s-s_0)t}\, dt.$$

Therefore, for $s > s_0$ we can take the limit as $N \to \infty$,

$$\big|\mathcal{L}[f]\big| \leqslant \lim_{N\to\infty} \Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant k\, \mathcal{L}[e^{s_0 t}] = \frac{k}{(s - s_0)}.$$

Therefore, the comparison test for improper integrals implies that the Laplace Transform $\mathcal{L}[f]$ exists at least for $s > s_0$, and it also holds that

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s - s_0}, \qquad s > s_0.$$

This establishes the Theorem. □

The next result says that the Laplace Transform is a linear transformation. This means that the Laplace Transform of a linear combination of functions is the linear combination of their Laplace Transforms.

Theorem 4.1.4 (Linear Combination). *If the Laplace transforms $\mathcal{L}[f]$ and $\mathcal{L}[g]$ of the functions f and g exist and a, b are constants, then the following equation holds*

$$\mathcal{L}[af + bg] = a\, \mathcal{L}[f] + b\, \mathcal{L}[g].$$

Proof of Theorem 4.1.4: Since integration is a linear operation, so is the Laplace Transform, as this calculation shows,

$$\begin{aligned}
\mathcal{L}[af + bg] &= \int_0^\infty e^{-st} \big[af(t) + bg(t)\big]\, dt \\
&= a \int_0^\infty e^{-st} f(t)\, dt + b \int_0^\infty e^{-st} g(t)\, dt \\
&= a\, \mathcal{L}[f] + b\, \mathcal{L}[g].
\end{aligned}$$

This establishes the Theorem. □

EXAMPLE 4.1.6: Compute $\mathcal{L}[3t^2 + 5\cos(4t)]$.

SOLUTION: From the Theorem above and the Laplace Transform in Table ?? we know that

$$\mathcal{L}[3t^2 + 5\cos(4t)] = 3\,\mathcal{L}[t^2] + 5\,\mathcal{L}[\cos(4t)]$$
$$= 3\Big(\frac{2}{s^3}\Big) + 5\Big(\frac{s}{s^2+4^2}\Big), \qquad s > 0$$
$$= \frac{6}{s^3} + \frac{5s}{s^2+4^2}.$$

Therefore,

$$\mathcal{L}[3t^2 + 5\cos(4t)] = \frac{5s^4 + 6s^2 + 96}{s^3(s^2+16)}, \qquad s > 0.$$

◁

The Laplace Transform can be used to solve differential equations. The Laplace Transform converts a differential equation into an algebraic equation. This is so because the Laplace Transform converts derivatives into multiplications. Here is the precise result.

Theorem 4.1.5 (Derivative). *If a function f is continuously differentiable on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}[f']$ exists for $s > s_0$ and*

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0. \tag{4.1.5}$$

Proof of Theorem 4.1.5: The main calculation in this proof is to compute

$$\mathcal{L}[f'] = \lim_{N\to\infty} \int_0^N e^{-st} f'(t)\,dt.$$

We start computing the definite integral above. Since f' is continuous on $[0,\infty)$, that definite integral exists for all positive N, and we can integrate by parts,

$$\int_0^N e^{-st} f'(t)\,dt = \Big[\Big(e^{-st} f(t)\Big)\Big|_0^N - \int_0^N (-s)e^{-st} f(t)\,dt\Big]$$
$$= e^{-sN} f(N) - f(0) + s\int_0^N e^{-st} f(t)\,dt.$$

We now compute the limit of this expression above as $N \to \infty$. Since f is continuous on $[0,\infty)$ of exponential order s_0, we know that

$$\lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt = \mathcal{L}[f], \qquad s > s_0.$$

Let us use one more time that f is of exponential order s_0. This means that there exist positive constants k and T such that $|f(t)| \leqslant k\,e^{s_0 t}$, for $t > T$. Therefore,

$$\lim_{N\to\infty} e^{-sN} f(N) \leqslant \lim_{N\to\infty} k\,e^{-sN} e^{s_0 N} = \lim_{N\to\infty} k\,e^{-(s-s_0)N} = 0, \qquad s > s_0.$$

These two results together imply that $\mathcal{L}[f']$ exists and holds

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0.$$

This establishes the Theorem. □

EXAMPLE 4.1.7: Verify the result in Theorem 4.1.5 for the function $f(t) = \cos(bt)$.

SOLUTION: We need to compute the left hand side and the right hand side of Eq. (4.1.5) and verify that we get the same result. We start with the left hand side,

$$\mathcal{L}[f'] = \mathcal{L}[-b\sin(bt)] = -b\,\mathcal{L}[\sin(bt)] = -b\,\frac{b}{s^2+b^2} \quad \Rightarrow \quad \mathcal{L}[f'] = -\frac{b^2}{s^2+b^2}.$$

We now compute the right hand side,

$$s\,\mathcal{L}[f] - f(0) = s\,\mathcal{L}[\cos(bt)] - 1 = s\,\frac{s}{s^2+b^2} - 1 = \frac{s^2 - s^2 - b^2}{s^2+b^2},$$

so we get

$$s\,\mathcal{L}[f] - f(0) = -\frac{b^2}{s^2+b^2}.$$

We conclude that $\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)$. ◁

It is not difficult to generalize Theorem 4.1.5 to higher order derivatives.

Theorem 4.1.6 (Higher Derivatives). *If a function f is n times continuously differentiable on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}[f''],\cdots,\mathcal{L}[f^{(n)}]$ exist for $s > s_0$ and*

$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) \tag{4.1.6}$$

$$\vdots$$

$$\mathcal{L}[f^{(n)}] = s^n\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0). \tag{4.1.7}$$

Proof of Theorem 4.1.6: We need to use Eq. (4.1.5) n times. We start with the Laplace Transform of a second derivative,

$$\begin{aligned}\mathcal{L}[f''] &= \mathcal{L}[(f')'] \\ &= s\,\mathcal{L}[f'] - f'(0) \\ &= s\big(s\,\mathcal{L}[f] - f(0)\big) - f'(0) \\ &= s^2\,\mathcal{L}[f] - s\,f(0) - f'(0).\end{aligned}$$

The formula for the Laplace Transform of an nth derivative is computed by induction on n. We assume that the formula is true for $n-1$,

$$\mathcal{L}[f^{(n-1)}] = s^{(n-1)}\,\mathcal{L}[f] - s^{(n-2)}\,f(0) - \cdots - f^{(n-2)}(0).$$

Since $\mathcal{L}[f^{(n)}] = \mathcal{L}[(f')^{(n-1)}]$, the formula above on f' gives

$$\begin{aligned}\mathcal{L}[(f')^{(n-1)}] &= s^{(n-1)}\,\mathcal{L}[f'] - s^{(n-2)}\,f'(0) - \cdots - (f')^{(n-2)}(0) \\ &= s^{(n-1)}\,\big(s\,\mathcal{L}[f] - f(0)\big) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0) \\ &= s^{(n)}\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0).\end{aligned}$$

This establishes the Theorem. □

EXAMPLE 4.1.8: Verify Theorem 4.1.6 for f'', where $f(t) = \cos(bt)$.

SOLUTION: We need to compute the left hand side and the right hand side in the first equation in Theorem (4.1.6), and verify that we get the same result. We start with the left hand side,

$$\mathcal{L}[f''] = \mathcal{L}[-b^2\cos(bt)] = -b^2\,\mathcal{L}[\cos(bt)] = -b^2\,\frac{s}{s^2+b^2} \quad\Rightarrow\quad \mathcal{L}[f''] = -\frac{b^2 s}{s^2+b^2}.$$

We now compute the right hand side,

$$s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) = s^2\,\mathcal{L}[\cos(bt)] - s - 0 = s^2\,\frac{s}{s^2+b^2} - s = \frac{s^3 - s^3 - b^2 s}{s^2+b^2},$$

so we get

$$s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) = -\frac{b^2 s}{s^2+b^2}.$$

We conclude that $\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0)$. ◁

4.1.4. **Exercises.**

4.1.1.- .

4.1.2.- .

4.2. The Initial Value Problem

4.2.1. **Solving Differential Equations.**

We plan to use the Laplace Transform to solve differential equations. Roughly, this is done as follows:

$$\mathcal{L}\begin{bmatrix}\text{differential eq.}\\ \text{for } y(t).\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{algebraic eq.}\\ \text{for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Use the table.)}\end{matrix}$$

Remarks:

(a) We will use the Laplace Transform to solve differential equations with *constant coefficients*. Although the method can be used with variable coefficients equations, the calculations could be very complicated in this case.

(b) The Laplace Transform method works with *very general source functions*, including step functions, which are discontinuous, and Dirac's deltas, which are generalized functions.

As we see in the sketch above, we start with a differential equation for a function y. We first compute the Laplace Transform of the whole differential equation. Then we use the linearity of the Laplace Transform, Theorem 4.1.4, and the property that derivatives are converted into multiplications, Theorem 4.1.5, to transform the differential equation into an algebraic equation for $\mathcal{L}[y]$. Let us see how this works in a simple example, a first order linear equation with constant coefficients. We learned how to solve such equation in § 1.1.

Example 4.2.1: Use the Laplace Transform to find the solution y to the initial value problem

$$y' + 2y = 0, \qquad y(0) = 3.$$

Solution: In § 1.1 we saw one way to solve this problem, using the integrating factor method. One can check that the solution is $y(t) = 3e^{-2t}$. We now use the Laplace Transform. First, compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[0] = 0.$$

Theorem 4.1.4 says the Laplace Transform is a linear operation, that is,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = 0.$$

Theorem 4.1.5 relates derivatives and multiplications, as follows,

$$\Big[s\,\mathcal{L}[y] - y(0)\Big] + 2\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s+2)\mathcal{L}[y] = y(0).$$

In the last equation we have been able to transform the original differential equation for y into an algebraic equation for $\mathcal{L}[y]$. We can solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{y(0)}{s+2} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{3}{s+2},$$

where in the last step we introduced the initial condition $y(0) = 3$. From the list of Laplace Transforms given in Sect. 4.1 we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{3}{s+2} = 3\mathcal{L}[e^{-2t}] \quad \Rightarrow \quad \frac{3}{s+2} = \mathcal{L}[3\,e^{-2t}].$$

So we arrive at $\mathcal{L}[y(t)] = \mathcal{L}[3\,e^{-2t}]$. Here is where we need one more property of the Laplace Transform. We show right after this example that

$$\mathcal{L}[y(t)] = \mathcal{L}[3\,e^{-2t}] \quad \Rightarrow \quad y(t) = 3\,e^{-2t}.$$

This property is called One-to-One. Hence the only solution is $y(t) = 3\,e^{-2t}$. ◁

4.2.2. **One-to-One Property.** Let us repeat the method we used to solve the differential equation in Example 4.2.1. We first computed the Laplace Transform of the whole differential equation. Then we use the linearity of the Laplace Transform, Theorem 4.1.4, and the property that derivatives are converted into multiplications, Theorem 4.1.5, to transform the differential equation into an algebraic equation for $\mathcal{L}[y]$. We solved the algebraic equation and we got an expression of the form

$$\mathcal{L}[y(t)] = H(s),$$

where we have collected all the terms that come from the Laplace transformed differential equation into the function H. We then used a Laplace Transform table to find a function h such that

$$\mathcal{L}[h(t)] = H(s).$$

We arrived to an equation of the form

$$\mathcal{L}[y(t)] = \mathcal{L}[h(t)].$$

Clearly, $y = h$ is one solution of the equation above, hence a solution to the differential equation. We now show that there are no solutions to the equation $\mathcal{L}[y] = \mathcal{L}[h]$ other than $y = h$. The reason is that the Laplace Transform on continuous functions of exponential order is an one-to-one transformation, also called injective.

Theorem 4.2.1 (One-to-One). *If f, g are continuous on $[0, \infty)$ of exponential order, then*

$$\mathcal{L}[f] = \mathcal{L}[g] \quad \Rightarrow \quad f = g.$$

Remarks:

(a) The result above holds for continuous functions f and g. But it can be extended to piecewise continuous functions. In the case of piecewise continuous functions f and g satisfying $\mathcal{L}[f] = \mathcal{L}[g]$ one can prove that $f = g + h$, where h is a null function, meaning that $\int_0^T h(t)\,dt = 0$ for all $T > 0$. See Churchill's textbook [4], page 14.

(b) Once we know that the Laplace Transform is a one-to-one transformation, we can define the inverse transformation in the usual way.

Definition 4.2.2. *The* ***Inverse Laplace Transform****, denoted $\mathcal{L}^{-1}$, of a function F is*

$$\mathcal{L}^{-1}[F(s)] = f(t) \quad \Leftrightarrow \quad F(s) = \mathcal{L}[f(t)].$$

Remarks: There is an explicit formula for the inverse Laplace Transform, which involves an integral on the complex plane,

$$\mathcal{L}^{-1}[F(s)]\Big|_t = \frac{1}{2\pi i} \lim_{c\to\infty} \int_{a-ic}^{a+ic} e^{st}\, F(s)\, ds.$$

See for example Churchill's textbook [4], page 176. However, we do not use this formula in these notes, since it involves integration on the complex plane.

Proof of Theorem 4.2.1: The proof is based on a clever change of variables and on Weierstrass Approximation Theorem of continuous functions by polynomials. Before we get to the change of variable we need to do some rewriting. Introduce the function $u = f - g$, then the linearity of the Laplace Transform implies

$$\mathcal{L}[u] = \mathcal{L}[f - g] = \mathcal{L}[f] - \mathcal{L}[g] = 0.$$

What we need to show is that the function u vanishes identically. Let us start with the definition of the Laplace Transform,

$$\mathcal{L}[u] = \int_0^\infty e^{-st}\, u(t)\, dt.$$

We know that f and g are of exponential order, say s_0, therefore u is of exponential order s_0, meaning that there exist positive constants k and T such that

$$\big|u(t)\big| < k\, e^{s_0 t}, \qquad t > T.$$

Evaluate $\mathcal{L}[u]$ at $\tilde{s} = s_1 + n + 1$, where s_1 is any real number such that $s_1 > s_0$, and n is any positive integer. We get

$$\mathcal{L}[u]\Big|_{\tilde{s}} = \int_0^\infty e^{-(s_1+n+1)t}\, u(t)\, dt = \int_0^\infty e^{-s_1 t}\, e^{-(n+1)t}\, u(t)\, dt.$$

We now do the substitution $y = e^{-t}$, so $dy = -e^{-t}\, dt$,

$$\mathcal{L}[u]\Big|_{\tilde{s}} = \int_1^0 y^{s_1}\, y^n\, u\big(-\ln(y)\big)\, (-dy) = \int_0^1 y^{s_1}\, y^n\, u\big(-\ln(y)\big)\, dy.$$

Introduce the function $v(y) = y^{s_1}\, u(\big(-\ln(y)\big)$, so the integral is

$$\mathcal{L}[u]\Big|_{\tilde{s}} = \int_0^1 y^n\, v(y)\, dy. \tag{4.2.1}$$

We know that $\mathcal{L}[u]$ exists because u is continuous and of exponential order, so the function v does not diverge at $y = 0$. To double check this, recall that $t = -\ln(y) \to \infty$ as $y \to 0^+$, and u is of exponential order s_0, hence

$$\lim_{y\to 0^+} |v(y)| = \lim_{t\to\infty} e^{-s_1 t}|u(t)| < \lim_{t\to\infty} e^{-(s_1 - s_0)t} = 0.$$

Our main hypothesis is that $\mathcal{L}[u] = 0$ for all values of s such that $\mathcal{L}[u]$ is defined, in particular $\tilde{s}$. By looking at Eq. (4.2.1) this means that

$$\int_0^1 y^n\, v(y)\, dy = 0, \qquad n = 1, 2, 3, \cdots .$$

The equation above and the linearity of the integral imply that this function v is perpendicular to every polynomial p, that is

$$\int_0^1 p(y)\, v(y)\, dy = 0, \tag{4.2.2}$$

for every polynomial p. Knowing that, we can do the following calculation,

$$\int_0^1 v^2(y)\, dy = \int_0^1 \big(v(y) - p(y)\big)\, v(y)\, dy + \int_0^1 p(y)\, v(y)\, dy.$$

The last term in the second equation above vanishes because of Eq. (4.2.2), therefore

$$\begin{aligned}
\int_0^1 v^2(y)\, dy &= \int_0^1 \big(v(y) - p(y)\big)\, v(y)\, dy \\
&\leqslant \int_0^1 \big|v(y) - p(y)\big|\, |v(y)|\, dy \\
&\leqslant \max_{y\in[0,1]} |v(y)| \int_0^1 \big|v(y) - p(y)\big|\, dy.
\end{aligned} \tag{4.2.3}$$

We remark that the inequality above is true for every polynomial p. Here is where we use the Weierstrass Approximation Theorem, which essentially says that every continuous function on a closed interval can be approximated by a polynomial.

Theorem 4.2.3 (Weierstrass Approximation). *If f is a continuous function on a closed interval $[a, b]$, then for every $\epsilon > 0$ there exists a polynomial q_ϵ such that*

$$\max_{y\in[a,b]} |f(y) - q_\epsilon(y)| < \epsilon.$$

The proof of this theorem can be found on a real analysis textbook. Weierstrass result implies that, given v and $\epsilon > 0$, there exists a polynomial p_ϵ such that the inequality in (4.2.3) has the form

$$\int_0^1 v^2(y)\,dy \leqslant \max_{y\in[0,1]} |v(y)| \int_0^1 \big|v(y) - p_\epsilon(y)\big|\,dy \leqslant \max_{y\in[0,1]} |v(y)|\,\epsilon.$$

Since ϵ can be chosen as small as we please, we get

$$\int_0^1 v^2(y)\,dy = 0.$$

But v is continuous, hence $v = 0$, meaning that $f = g$. This establishes the Theorem. □

4.2.3. **Partial Fractions.** We are now ready to start using the Laplace Transform to solve second order linear differential equations with constant coefficients. The differential equation for y will be transformed into an algebraic equation for $\mathcal{L}[y]$. We will then arrive to an equation of the form $\mathcal{L}[y(t)] = H(s)$. We will see, already in the first example below, that usually this function H does not appear in Table 2. We will need to rewrite H as a linear combination of simpler functions, each one appearing in Table 2. One of the more used techniques to do that is called Partial Fractions. Let us solve the next example.

EXAMPLE 4.2.2: Use the Laplace Transform to find the solution y to the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0.$$

SOLUTION: First, compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0] = 0.$$

Theorem 4.1.4 says that the Laplace Transform is a linear operation,

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\,\mathcal{L}[y] = 0.$$

Then, Theorem 4.1.5 relates derivatives and multiplications,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - \Big[s\,\mathcal{L}[y] - y(0)\Big] - 2\,\mathcal{L}[y] = 0,$$

which is equivalent to the equation

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1)\,y(0) + y'(0).$$

Once again we have transformed the original differential equation for y into an algebraic equation for $\mathcal{L}[y]$. Introduce the initial condition into the last equation above, that is,

$$(s^2 - s - 2)\,\mathcal{L}[y] = (s-1).$$

Solve for the unknown $\mathcal{L}[y]$ as follows,

$$\mathcal{L}[y] = \frac{(s-1)}{(s^2 - s - 2)}.$$

The function on the right hand side above does not appear in Table 2. We now use *partial fractions* to find a function whose Laplace Transform is the right hand side of the equation above. First find the roots of the polynomial in the denominator,

$$s^2 - s - 2 = 0 \quad\Rightarrow\quad s_\pm = \frac{1}{2}\big[1 \pm \sqrt{1+8}\big] \quad\Rightarrow\quad \begin{cases} s_+ = 2, \\ s_- = -1, \end{cases}$$

that is, the polynomial has two real roots. In this case we factorize the denominator,

$$\mathcal{L}[y] = \frac{(s-1)}{(s-2)(s+1)}.$$

The partial fraction decomposition of the right-hand side in the equation above is the following: Find constants a and b such that

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}.$$

A simple calculation shows

$$\frac{(s-1)}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)} = \frac{s(a+b)+(a-2b)}{(s-2)(s+1)}.$$

Hence constants a and b must be solutions of the equations

$$(s-1) = s(a+b) + (a-2b) \quad \Rightarrow \quad \begin{cases} a+b=1, \\ a-2b=-1. \end{cases}$$

The solution is $a = \dfrac{1}{3}$ and $b = \dfrac{2}{3}$. Hence,

$$\mathcal{L}[y] = \frac{1}{3}\,\frac{1}{(s-2)} + \frac{2}{3}\,\frac{1}{(s+1)}.$$

From the list of Laplace Transforms given in § 4.1, Table 2, we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}], \qquad \frac{1}{s+1} = \mathcal{L}[e^{-t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \frac{1}{3}\,\mathcal{L}[e^{2t}] + \frac{2}{3}\,\mathcal{L}[e^{-t}] = \mathcal{L}\Big[\frac{1}{3}\big(e^{2t} + 2e^{-t}\big)\Big]$$

We conclude that

$$y(t) = \frac{1}{3}\big(e^{2t} + 2e^{-t}\big).$$

◁

The Partial Fraction Method is usually introduced in a second course of Calculus to integrate rational functions. We need it here to use Table 2 to find Inverse Laplace Transforms. The method applies to rational functions

$$R(s) = \frac{Q(s)}{P(s)},$$

where P, Q are polynomials and the degree of the numerator is less than the degree of the denominator. In the example above

$$R(s) = \frac{(s-1)}{(s^2-s-2)}.$$

One starts rewriting the polynomial in the denominator as a product of polynomials degree two or one. In the example above,

$$R(s) = \frac{(s-1)}{(s-2)(s+1)}.$$

One then rewrites the rational function as an addition of simpler rational functions. In the example above,

$$R(s) = \frac{a}{(s-2)} + \frac{b}{(s+1)}.$$

We now solve a few examples to recall the different partial fraction cases that can appear when solving differential equations.

EXAMPLE 4.2.3: Use the Laplace Transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

SOLUTION: First, compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[0] = 0.$$

Theorem 4.1.4 says that the Laplace Transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = 0.$$

Theorem 4.1.5 relates derivatives with multiplication,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = 0,$$

which is equivalent to the equation

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s-4)\,y(0) + y'(0).$$

Introduce the initial conditions $y(0) = 1$ and $y'(0) = 1$ into the equation above,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3.$$

Solve the algebraic equation for $\mathcal{L}[y]$,

$$\mathcal{L}[y] = \frac{(s-3)}{(s^2 - 4s + 4)}.$$

We now want to find a function y whose Laplace Transform is the right hand side in the equation above. In order to see if partial fractions will be needed, we now find the roots of the polynomial in the denominator,

$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-16}\big] \quad \Rightarrow \quad s_+ = s_- = 2.$$

that is, the polynomial has a single real root, so $\mathcal{L}[y]$ can be written as

$$\mathcal{L}[y] = \frac{(s-3)}{(s-2)^2}.$$

This expression is already in the partial fraction decomposition. We now rewrite the right hand side of the equation above in a way it is simple to use the Laplace Transform table in § 4.1,

$$\mathcal{L}[y] = \frac{(s-2) + 2 - 3}{(s-2)^2} = \frac{(s-2)}{(s-2)^2} - \frac{1}{(s-2)^2} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{1}{s-2} - \frac{1}{(s-2)^2}.$$

From the list of Laplace Transforms given in Table 2, § 4.1 we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2} \quad \Rightarrow \quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the equation

$$\mathcal{L}[y] = \mathcal{L}[e^{2t}] - \mathcal{L}[te^{2t}] = \mathcal{L}\big[e^{2t} - te^{2t}\big] \quad \Rightarrow \quad y(t) = e^{2t} - te^{2t}.$$

◁

EXAMPLE 4.2.4: Use the Laplace Transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

SOLUTION: First, compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\sin(2t)].$$

The right hand side above can be expressed as follows,

$$\mathcal{L}[3\sin(2t)] = 3\,\mathcal{L}[\sin(2t)] = 3\,\frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.$$

Theorem 4.1.4 says that the Laplace Transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4},$$

and Theorem 4.1.5 relates derivatives with multiplications,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\,\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

Reorder terms,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s - 4)\,y(0) + y'(0) + \frac{6}{s^2 + 4}.$$

Introduce the initial conditions $y(0) = 1$ and $y'(0) = 1$,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}.$$

Solve this algebraic equation for $\mathcal{L}[y]$, that is,

$$\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4 + 4)(s^2 + 4)}.$$

From the Example above we know that $s^2 - 4s + 4 = (s - 2)^2$, so we obtain

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} + \frac{6}{(s - 2)^2(s^2 + 4)}. \tag{4.2.4}$$

From the previous example we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}. \tag{4.2.5}$$

We know use *partial fractions* to simplify the third term on the right hand side of Eq. (4.2.4). The appropriate partial fraction decomposition for this term is the following: Find constants a, b, c, d, such that

$$\frac{6}{(s - 2)^2\,(s^2 + 4)} = \frac{as + b}{s^2 + 4} + \frac{c}{(s - 2)} + \frac{d}{(s - 2)^2}$$

Take common denominator on the right hand side above, and one obtains the system

$$\begin{aligned} a + c &= 0, \\ -4a + b - 2c + d &= 0, \\ 4a - 4b + 4c &= 0, \\ 4b - 8c + 4d &= 6. \end{aligned}$$

The solution for this linear system of equations is the following:

$$a = \frac{3}{8}, \qquad b = 0, \qquad c = -\frac{3}{8}, \qquad d = \frac{3}{4}.$$

Therefore,

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\frac{s}{s^2+4} - \frac{3}{8}\,\frac{1}{(s-2)} + \frac{3}{4}\,\frac{1}{(s-2)^2}$$

We can rewrite this expression above in terms of the Laplace Transforms given in Table 2, in Sect. 4.1, as follows,

$$\frac{6}{(s-2)^2\,(s^2+4)} = \frac{3}{8}\,\mathcal{L}[\cos(2t)] - \frac{3}{8}\,\mathcal{L}[e^{2t}] + \frac{3}{4}\,\mathcal{L}[te^{2t}],$$

and using the linearity of the Laplace Transform,

$$\frac{6}{(s-2)^2\,(s^2+4)} = \mathcal{L}\Big[\frac{3}{8}\,\cos(2t) - \frac{3}{8}\,e^{2t} + \frac{3}{4}\,te^{2t}\Big]. \tag{4.2.6}$$

Finally, introducing Eqs. (4.2.5) and (4.2.6) into Eq. (4.2.4) we obtain

$$\mathcal{L}[y(t)] = \mathcal{L}\Big[(1-t)\,e^{2t} + \frac{3}{8}\,(-1+2t)\,e^{2t} + \frac{3}{8}\,\cos(2t)\Big].$$

Since the Laplace Transform is an invertible transformation, we conclude that

$$y(t) = (1-t)\,e^{2t} + \frac{3}{8}\,(2t-1)\,e^{2t} + \frac{3}{8}\,\cos(2t).$$

$\lhd$

4.2.4. **Exercises.**

4.2.1.- .

4.2.2.- .

4.3. Discontinuous Sources

The Laplace Transform Method is useful to solve linear differential equations with discontinuous source functions. In this section review what could be the simplest discontinuous function, the step function, and we use it to construct more general piecewise continuous functions. We then compute the Laplace Transform of these discontinuous functions. We also find formulas for the Laplace Transform of certain translations of functions. These formulas and the Laplace Transform Table in Section 4.1 are very important to solve differential equations with discontinuous sources.

4.3.1. Step Functions.

We start with a definition of a step function.

Definition 4.3.1. *The* ***step function*** *at $t = 0$ is denoted by u and given by*

$$u(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geqslant 0. \end{cases} \tag{4.3.1}$$

The step function u at $t = 0$ and its right and left translations are plotted in Fig. 15.

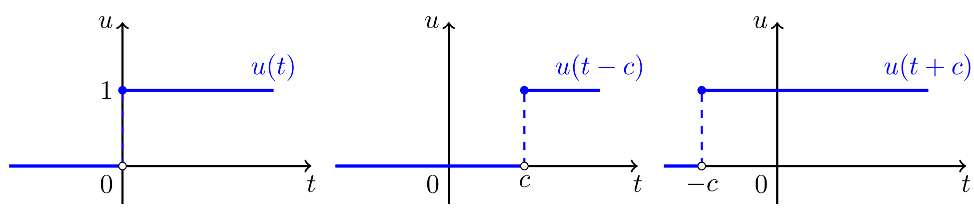


FIGURE 15. The graph of the step function given in Eq. (4.3.1), a right and a left translation by a constant $c > 0$, respectively, of this step function.

Recall that given a function with values $f(t)$ and a positive constant c, then $f(t-c)$ and $f(t+c)$ are the function values of the right translation and the left translation, respectively, of the original function f. In Fig. 16 we plot the graph of functions $f(t) = e^{at}$, $g(t) = u(t)\, e^{at}$ and their respective right translations by $c > 0$.

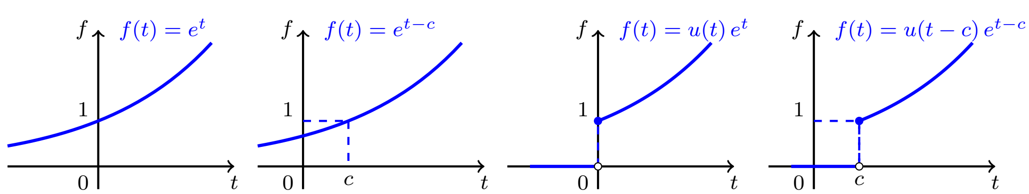


FIGURE 16. The function $f(t) = e^t$, its right translation by $c > 0$, the function $f(t) = u(t)\, e^{at}$ and its right translation by c.

Right and left translations of step functions are useful to construct bump functions. A bump function is a function with nonzero values only on a finite interval $[a, b]$.

$$b(t) = u(t-a) - u(t-b) \quad \Leftrightarrow \quad b(t) = \begin{cases} 0 & t < a, \\ 1 & a \leqslant t < b \\ 0 & t \geqslant b. \end{cases} \tag{4.3.2}$$

The graph of a bump function is given in Fig. 17, constructed from two step functions. Step and bump functions are useful to construct more general piecewise continuous functions.

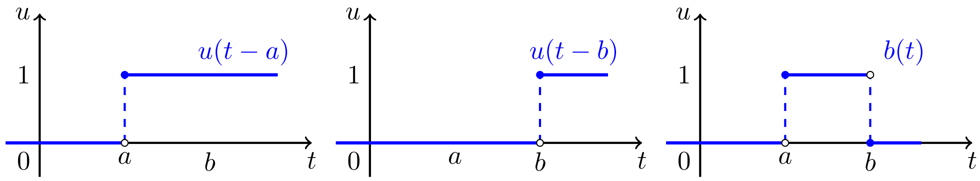


Figure 17. A bump function b constructed with translated step functions.

Example 4.3.1: Graph the function

$$f(t) = \big[u(t-1) - u(t-2)\big]\, e^{at}.$$

Solution: Recall that the function

$$b(t) = u(t-1) - u(t-2),$$

is a bump function with sides at $t = 1$ andf $t = 2$. Then, the function

$$f(t) = b(t)\, e^{at},$$

is nonzero where b is nonzero, that is on $[1, 2)$, and on that domain it takes values e^{at}. The graph of f is given in Fig. 18. ◁

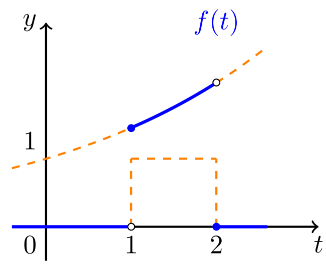


Figure 18. Function f.

The Laplace Transform of step functions are not difficult to compute.

Theorem 4.3.2. *For every number $c \in \mathbb{R}$ and and every $s > 0$ holds*

$$\mathcal{L}[u(t-c)] = \begin{cases} \dfrac{e^{-cs}}{s} & \text{for} \quad c \geqslant 0, \\[2mm] \dfrac{1}{s} & \text{for} \quad c < 0. \end{cases}$$

Proof of Theorem 4.3.2: Consider the case $c \geqslant 0$. The Laplace Transform is

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt = \int_c^\infty e^{-st}\, dt,$$

where we used that the step function vanishes for $t < c$. Now compute the improper integral,

$$\mathcal{L}[u(t-c)] = \lim_{N\to\infty} -\frac{1}{s}\big(e^{-Ns} - e^{-cs}\big) = \frac{e^{-cs}}{s} \quad \Rightarrow \quad \mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}.$$

Consider now the case of $c < 0$. The step function is identically equal to one in the domain of integration of the Laplace Transform, which is $[0, \infty)$, hence

$$\mathcal{L}[u(t-c)] = \int_0^\infty e^{-st} u(t-c)\, dt = \int_0^\infty e^{-st}\, dt = \mathcal{L}[1] = \frac{1}{s}.$$

This establishes the Theorem. □

Example 4.3.2: Compute $\mathcal{L}[3u(t-2)]$.

Solution: The Laplace Transform is a linear operation, so

$$\mathcal{L}[3u(t-2)] = 3\, \mathcal{L}[u(t-2)],$$

and the Theorem 4.3.2 above implies that $\mathcal{L}[3u(t-2)] = \dfrac{3\, e^{-2s}}{s}$. ◁

EXAMPLE 4.3.3: Compute $\mathcal{L}^{-1}\Big[\frac{e^{-3s}}{s}\Big]$.

SOLUTION: Theorem 4.3.2 says that $\frac{e^{-3s}}{s} = \mathcal{L}[u(t-3)]$, so $\mathcal{L}^{-1}\Big[\frac{e^{-3s}}{s}\Big] = u(t-3)$. ◁

4.3.2. **Translation Identities.** We now introduce two important properties of the Laplace Transform.

Theorem 4.3.3 (Translation Identities). *If $\mathcal{L}[f(t)](s)$ exists for $s > a$, then*

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)], \qquad s > a, \qquad c \geqslant 0 \tag{4.3.3}$$

$$\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)](s-c), \qquad s > a+c, \qquad c \in \mathbb{R}. \tag{4.3.4}$$

Remarks:

(a) Eq. (4.3.4) holds for all $c \in \mathbb{R}$, while Eq. (4.3.3) holds only for $c \geqslant 0$.

(b) Show that in the case that $c < 0$ the following equation holds,

$$\mathcal{L}[u(t+|c|)f(t+|c|)] = e^{|c|s}\Big(\mathcal{L}[f(t)] - \int_0^{|c|} e^{-st}\,f(t)\,dt\Big).$$

(c) We can highlight the main idea in the theorem above as follows:

$$\mathcal{L}\big[\text{right-translation } (uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big),$$
$$\mathcal{L}\big[(\exp)\,(f)\big] = \text{translation}\big(\mathcal{L}[f]\big).$$

(d) Denoting $F(s) = \mathcal{L}[f(t)]$, then an equivalent expression for Eqs. (4.3.3)-(4.3.4) is

$$\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,F(s),$$
$$\mathcal{L}[e^{ct} f(t)] = F(s-c).$$

(e) The inverse form of Eqs. (4.3.3)-(4.3.4) is given by,

$$\mathcal{L}^{-1}[e^{-cs}\,F(s)] = u(t-c)f(t-c), \tag{4.3.5}$$

$$\mathcal{L}^{-1}[F(s-c)] = e^{ct} f(t). \tag{4.3.6}$$

Proof of Theorem 4.3.3: The proof is again based in a change of the integration variable. We start with Eq. (4.3.3), as follows,

$$\begin{aligned}
\mathcal{L}[u(t-c)f(t-c)] &= \int_0^\infty e^{-st}u(t-c)f(t-c)\,dt \\
&= \int_c^\infty e^{-st} f(t-c)\,dt, \quad \tau = t-c, \quad d\tau = dt, \quad c \geqslant 0, \\
&= \int_0^\infty e^{-s(\tau+c)} f(\tau)\,d\tau \\
&= e^{-cs} \int_0^\infty e^{-s\tau} f(\tau)\,d\tau \\
&= e^{-cs}\,\mathcal{L}[f(t)], \qquad s > a.
\end{aligned}$$

The proof of Eq. (4.3.4) is a bit simpler, since

$$\mathcal{L}\big[e^{ct} f(t)\big] = \int_0^\infty e^{-st}e^{ct} f(t)\,dt = \int_0^\infty e^{-(s-c)t} f(t)\,dt = \mathcal{L}[f(t)](s-c),$$

which holds for $s - c > a$. This establishes the Theorem. □

Example 4.3.4: Compute $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big]$.

Solution: Both $\mathcal{L}[\sin(at)] = \dfrac{a}{s^2+a^2}$ and $\mathcal{L}[u(t-c)f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$ imply

$$\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = e^{-2s}\,\mathcal{L}[\sin(at)] = e^{-2s}\,\frac{a}{s^2+a^2}.$$

We conclude: $\mathcal{L}\big[u(t-2)\,\sin(a(t-2))\big] = \dfrac{a\,e^{-2s}}{s^2+a^2}$. ◁

Example 4.3.5: Compute $\mathcal{L}\big[e^{3t}\,\sin(at)\big]$.

Solution: Since $\mathcal{L}[e^{ct}f(t)] = \mathcal{L}[f](s-c)$, then we get

$$\mathcal{L}\big[e^{3t}\,\sin(at)\big] = \frac{a}{(s-3)^2+a^2}, \qquad s>3.$$

◁

Example 4.3.6: Compute both $\mathcal{L}\big[u(t-2)\,\cos\big(a(t-2)\big)\big]$ and $\mathcal{L}\big[e^{3t}\,\cos(at)\big]$.

Solution: Since $\mathcal{L}\big[\cos(at)\big] = \dfrac{s}{s^2+a^2}$, then

$$\mathcal{L}\big[u(t-2)\,\cos\big(a(t-2)\big)\big] = e^{-2s}\,\frac{s}{(s^2+a^2)}, \qquad \mathcal{L}\big[e^{3t}\,\cos(at)\big] = \frac{(s-3)}{(s-3)^2+a^2}.$$

◁

Example 4.3.7: Find the Laplace Transform of the function

$$f(t) = \begin{cases} 0 & t<1, \\ (t^2-2t+2) & t \geqslant 1. \end{cases} \tag{4.3.7}$$

Solution: The idea is to rewrite function f so we can use the Laplace Transform Table 2, in § 4.1 to compute its Laplace Transform. Since the function f vanishes for all $t<1$, we use step functions to write f as

$$f(t) = u(t-1)(t^2-2t+2).$$

Now, notice that completing the square we obtain,

$$t^2-2t+2 = (t^2-2t+1)-1+2 = (t-1)^2+1.$$

The polynomial is a parabola t^2 translated to the right and up by one. This is a discontinuous function, as it can be seen in Fig. 19.

So the function f can be written as follows,

$$f(t) = u(t-1)\,(t-1)^2 + u(t-1).$$

Since we know that $\mathcal{L}[t^2] = \dfrac{2}{s^3}$, then Eq. (4.3.3) implies

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1)\,(t-1)^2] + \mathcal{L}[u(t-1)]$$
$$= e^{-s}\,\frac{2}{s^3} + e^{-s}\,\frac{1}{s}$$

so we get

$$\mathcal{L}[f(t)] = \frac{e^{-s}}{s^3}\big(2+s^2\big).$$

◁

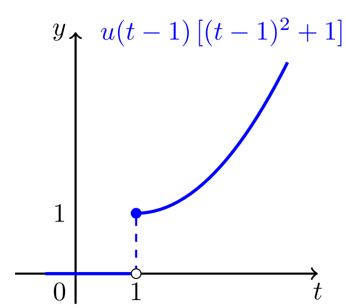


Figure 19. Function f given in Eq. (4.3.7).

EXAMPLE 4.3.8: Find the function f such that $\mathcal{L}[f(t)] = \dfrac{e^{-4s}}{s^2+5}$.

SOLUTION: Notice that

$$\mathcal{L}[f(t)] = e^{-4s}\,\frac{1}{s^2+5} \quad\Rightarrow\quad \mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,e^{-4s}\,\frac{\sqrt{5}}{s^2+\big(\sqrt{5}\big)^2}.$$

Recall that $\mathcal{L}[\sin(at)] = \dfrac{a}{(s^2+a^2)}$, then Eq. (4.3.3), or its inverse form Eq. (4.3.5) imply

$$\mathcal{L}[f(t)] = \frac{1}{\sqrt{5}}\,\mathcal{L}\big[u(t-4)\sin\big(\sqrt{5}\,(t-4)\big)\big] \quad\Rightarrow\quad f(t) = \frac{1}{\sqrt{5}}\,u(t-4)\sin\big(\sqrt{5}\,(t-4)\big).$$

◁

EXAMPLE 4.3.9: Find the function $f(t)$ such that $\mathcal{L}[f(t)] = \dfrac{(s-1)}{(s-2)^2+3}$.

SOLUTION: We first rewrite the right-hand side above as follows,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{(s-1-1+1)}{(s-2)^2+3}\\ &= \frac{(s-2)}{(s-2)^2+3} + \frac{1}{(s-2)^2+3}\\ &= \frac{(s-2)}{(s-2)^2+\big(\sqrt{3}\big)^2} + \frac{1}{\sqrt{3}}\,\frac{\sqrt{3}}{(s-2)^2+\big(\sqrt{3}\big)^2}.\end{aligned}$$

We now recall Eq. (4.3.4) or its inverse form Eq. (4.3.6), which imply

$$\mathcal{L}[f(t)] = \mathcal{L}\big[e^{2t}\cos\big(\sqrt{3}\,t\big)\big] + \frac{1}{\sqrt{3}}\mathcal{L}\big[e^{2t}\sin\big(\sqrt{3}\,t\big)\big].$$

So, we conclude that

$$f(t) = \frac{e^{2t}}{\sqrt{3}}\Big[\sqrt{3}\,\cos\big(\sqrt{3}\,t\big) + \sin\big(\sqrt{3}\,t\big)\Big].$$

◁

EXAMPLE 4.3.10: Find $\mathcal{L}^{-1}\Big[\dfrac{2e^{-3s}}{s^2-4}\Big]$.

SOLUTION: Since $\mathcal{L}^{-1}\Big[\dfrac{a}{s^2-a^2}\Big] = \sinh(at)$ and $\mathcal{L}^{-1}\big[e^{-cs}\,\hat{f}(s)\big] = u(t-c)\,f(t-c)$, then

$$\mathcal{L}^{-1}\Big[\frac{2e^{-3s}}{s^2-4}\Big] = \mathcal{L}^{-1}\Big[e^{-3s}\,\frac{2}{s^2-4}\Big] \quad\Rightarrow\quad \mathcal{L}^{-1}\Big[\frac{2e^{-3s}}{s^2-4}\Big] = u(t-3)\,\sinh\big(2(t-3)\big).$$

◁

EXAMPLE 4.3.11: Find a function f such that $\mathcal{L}[f(t)] = \dfrac{e^{-2s}}{s^2+s-2}$.

SOLUTION: Since the right hand side above does not appear in the Laplace Transform Table in § 4.1, we need to simplify it in an appropriate way. The plan is to rewrite the denominator of the rational function $1/(s^2+s-2)$, so we can use partial fractions to simplify this rational function. We first find out whether this denominator has real or complex roots:

$$s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1+8}\big] \quad\Rightarrow\quad \begin{cases} s_+ = 1,\\ s_- = -2.\end{cases}$$

We are in the case of real roots, so we rewrite

$$s^2 + s - 2 = (s-1)\,(s+2).$$

The partial fraction decomposition in this case is given by

$$\frac{1}{(s-1)\,(s+2)} = \frac{a}{(s-1)} + \frac{b}{(s+2)} = \frac{(a+b)\,s + (2a-b)}{(s-1)\,(s+2)} \quad \Rightarrow \quad \begin{cases} a+b=0, \\ 2a-b=1. \end{cases}$$

The solution is $a = 1/3$ and $b = -1/3$, so we arrive to the expression

$$\mathcal{L}[f(t)] = \frac{1}{3}\,e^{-2s}\,\frac{1}{s-1} - \frac{1}{3}\,e^{-2s}\,\frac{1}{s+2}.$$

Recalling that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a},$$

and Eq. (4.3.3) we obtain the equation

$$\mathcal{L}[f(t)] = \frac{1}{3}\,\mathcal{L}\big[u(t-2)\,e^{(t-2)}\big] - \frac{1}{3}\,\mathcal{L}\big[u(t-2)\,e^{-2(t-2)}\big]$$

which leads to the conclusion:

$$f(t) = \frac{1}{3}\,u(t-2)\Big[e^{(t-2)} - e^{-2(t-2)}\Big].$$

◁

4.3.3. **Solving Differential Equations.** The last three examples in this section show how to use the methods presented above to solve differential equations with discontinuous source functions.

Example 4.3.12: Use the Laplace Transform to find the solution of the initial value problem

$$y' + 2y = u(t-4), \qquad y(0) = 3.$$

Solution: We compute the Laplace Transform of the whole equation,

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[u(t-4)] = \frac{e^{-4s}}{s}.$$

From the previous section we know that

$$\big[s\,\mathcal{L}[y] - y(0)\big] + 2\,\mathcal{L}[y] = \frac{e^{-4s}}{s} \quad \Rightarrow \quad (s+2)\,\mathcal{L}[y] = y(0) + \frac{e^{-4s}}{s}.$$

We introduce the initial condition $y(0) = 3$ into equation above,

$$\mathcal{L}[y] = \frac{3}{(s+2)} + e^{-4s}\,\frac{1}{s(s+2)} \quad \Rightarrow \quad \mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + e^{-4s}\,\frac{1}{s(s+2)}.$$

We need to invert the Laplace Transform on the last term on the right hand side in equation above. We use the partial fraction decomposition on the rational function above, as follows

$$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{(s+2)} = \frac{a(s+2) + bs}{s(s+2)} = \frac{(a+b)\,s + (2a)}{s(s+2)} \quad \Rightarrow \quad \begin{cases} a+b=0, \\ 2a=1. \end{cases}$$

We conclude that $a = 1/2$ and $b = -1/2$, so

$$\frac{1}{s(s+2)} = \frac{1}{2}\Big[\frac{1}{s} - \frac{1}{(s+2)}\Big].$$

We then obtain

$$\mathcal{L}[y] = 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big[e^{-4s}\,\frac{1}{s} - e^{-4s}\,\frac{1}{(s+2)}\Big]$$
$$= 3\,\mathcal{L}\big[e^{-2t}\big] + \frac{1}{2}\Big(\mathcal{L}[u(t-4)] - \mathcal{L}\big[u(t-4)\,e^{-2(t-4)}\big]\Big).$$

Hence, we conclude that

$$y(t) = 3e^{-2t} + \frac{1}{2}\,u(t-4)\Big[1 - e^{-2(t-4)}\Big].$$

◁

Example 4.3.13: Use the Laplace Transform to find the solution to the initial value problem

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad b(t) = \begin{cases} 1 & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{4.3.8}$$

Solution: From Fig. 20, the source function b can be written as

$$b(t) = u(t) - u(t-\pi).$$

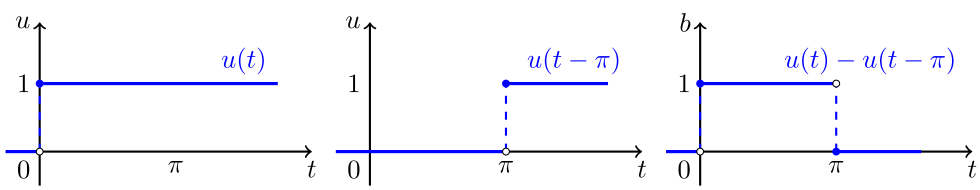


Figure 20. The graph of the u, its translation and b as given in Eq. (4.3.8).

The last expression for b is particularly useful to find its Laplace Transform,

$$\mathcal{L}[b(t)] = \mathcal{L}[u(t)] - \mathcal{L}[u(t-\pi)] = \frac{1}{s} + e^{-\pi s}\,\frac{1}{s} \quad \Rightarrow \quad \mathcal{L}[b(t)] = (1 - e^{-\pi s})\,\frac{1}{s}.$$

Now Laplace Transform the whole equation,

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[b].$$

Since the initial condition are $y(0) = 0$ and $y'(0) = 0$, we obtain

$$\Big(s^2 + s + \frac{5}{4}\Big)\,\mathcal{L}[y] = \big(1 - e^{-\pi s}\big)\,\frac{1}{s} \quad \Rightarrow \quad \mathcal{L}[y] = \big(1 - e^{-\pi s}\big)\,\frac{1}{s\,\Big(s^2 + s + \frac{5}{4}\Big)}.$$

Introduce the function

$$H(s) = \frac{1}{s\,\Big(s^2 + s + \frac{5}{4}\Big)} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}[H(s)] - \mathcal{L}^{-1}[e^{-\pi s}\,H(s)].$$

That is, we only need to find the inverse Laplace Transform of H. We use partial fractions to simplify the expression of H. We first find out whether the denominator has real or complex roots:

$$s^2 + s + \frac{5}{4} = 0 \quad \Rightarrow \quad s_{\pm} = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big],$$

so the roots are complex valued. An appropriate partial fraction decomposition is

$$H(s) = \frac{1}{s\,\big(s^2 + s + \frac{5}{4}\big)} = \frac{a}{s} + \frac{(bs + c)}{\big(s^2 + s + \frac{5}{4}\big)}$$

Therefore, we get

$$1 = a\Big(s^2 + s + \frac{5}{4}\Big) + s\,(bs + c) = (a+b)\,s^2 + (a+c)\,s + \frac{5}{4}a.$$

This equation implies that a, b, and c, satisfy the equations

$$a + b = 0, \quad a + c = 0, \quad \frac{5}{4}\,a = 1.$$

The solution is, $a = \frac{4}{5}$, $b = -\frac{4}{5}$, $c = -\frac{4}{5}$. Hence, we have found that,

$$H(s) = \frac{1}{\big(s^2 + s + \frac{5}{4}\big)\,s} = \frac{4}{5}\Big[\frac{1}{s} - \frac{(s+1)}{\big(s^2 + s + \frac{5}{4}\big)}\Big]$$

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)\,s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

Replace this expression in the definition of H, that is,

$$H(s) = \frac{4}{5}\Big[\frac{1}{s} - \frac{(s+1)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big]$$

Rewrite the polynomial in the numerator,

$$(s+1) = \Big(s + \frac{1}{2} + \frac{1}{2}\Big) = \Big(s + \frac{1}{2}\Big) + \frac{1}{2},$$

hence we get

$$H(s) = \frac{4}{5}\Big[\frac{1}{s} - \frac{\big(s + \frac{1}{2}\big)}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]} - \frac{1}{2}\,\frac{1}{\big[\big(s + \frac{1}{2}\big)^2 + 1\big]}\Big].$$

Use the Laplace Transform table to get $H(s)$ equal to

$$H(s) = \frac{4}{5}\Big[\mathcal{L}[1] - \mathcal{L}\big[e^{-t/2}\cos(t)\big] - \frac{1}{2}\mathcal{L}[e^{-t/2}\sin(t)]\Big],$$

equivalently

$$H(s) = \mathcal{L}\Big[\frac{4}{5}\Big(1 - e^{-t/2}\cos(t) - \frac{1}{2}e^{-t/2}\sin(t)\Big)\Big].$$

Denote

$$h(t) = \frac{4}{5}\Big[1 - e^{-t/2}\cos(t) - \frac{1}{2}e^{-t/2}\sin(t)\Big]. \quad \Rightarrow \quad H(s) = \mathcal{L}[h(t)].$$

Recalling $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$, we obtain $\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)]$, that is,

$$y(t) = h(t) + u(t-\pi)h(t-\pi).$$

◁

Example 4.3.14: Use the Laplace Transform to find the solution to the initial value problem

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad g(t) = \begin{cases} \sin(t) & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{4.3.9}$$

Solution: From Fig. 21, the source function g can be written as the following product,

$$g(t) = \big[u(t) - u(t-\pi)\big]\,\sin(t),$$

since $u(t) - u(t-\pi)$ is a box function, taking value one in the interval $[0,\pi]$ and zero on the complement. Finally, notice that the equation $\sin(t) = -\sin(t-\pi)$ implies that the function g can be expressed as follows,

$$g(t) = u(t)\,\sin(t) - u(t-\pi)\,\sin(t) \quad \Rightarrow \quad g(t) = u(t)\,\sin(t) + u(t-\pi)\,\sin(t-\pi).$$

The last expression for g is particularly useful to find its Laplace Transform,

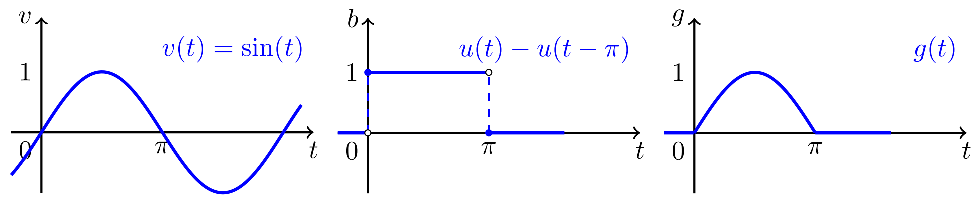


FIGURE 21. The graph of the sine function, a square function $u(t)-u(t-\pi)$ and the source function g given in Eq. (4.3.9).

$$\mathcal{L}[g(t)] = \frac{1}{(s^2+1)} + e^{-\pi s}\,\frac{1}{(s^2+1)}.$$

With this last transform is not difficult to solve the differential equation. As usual, Laplace Transform the whole equation,

$$\mathcal{L}[y''] + \mathcal{L}[y'] + \frac{5}{4}\,\mathcal{L}[y] = \mathcal{L}[g].$$

Since the initial condition are $y(0) = 0$ and $y'(0) = 0$, we obtain

$$\Big(s^2 + s + \frac{5}{4}\Big)\,\mathcal{L}[y] = \big(1 + e^{-\pi s}\big)\,\frac{1}{(s^2+1)} \quad \Rightarrow \quad \mathcal{L}[y] = \big(1 + e^{-\pi s}\big)\,\frac{1}{\Big(s^2 + s + \frac{5}{4}\Big)\,(s^2+1)}.$$

Introduce the function

$$H(s) = \frac{1}{\Big(s^2 + s + \frac{5}{4}\Big)\,(s^2+1)} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}[H(s)] + \mathcal{L}^{-1}[e^{-\pi s}\,H(s)].$$

That is, we only need to find the Inverse Laplace Transform of H. We use partial fractions to simplify the expression of H. We first find out whether the denominator has real or complex roots:

$$s^2 + s + \frac{5}{4} = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[-1 \pm \sqrt{1-5}\big],$$

so the roots are complex valued. An appropriate partial fraction decomposition is

$$H(s) = \frac{1}{\big(s^2 + s + \frac{5}{4}\big)\,(s^2+1)} = \frac{(as+b)}{\big(s^2 + s + \frac{5}{4}\big)} + \frac{(cs+d)}{(s^2+1)}.$$

Therefore, we get

$$1 = (as+b)(s^2+1) + (cs+d)\Big(s^2 + s + \frac{5}{4}\Big),$$

equivalently,

$$1 = (a+c)\,s^3 + (b+c+d)\,s^2 + \Big(a + \frac{5}{4}\,c + d\Big)\,s + \Big(b + \frac{5}{4}\,d\Big).$$

This equation implies that a, b, c, and d, are solutions of

$$a + c = 0, \quad b + c + d = 0, \quad a + \frac{5}{4}\,c + d = 0, \quad b + \frac{5}{4}\,d = 1.$$

Here is the solution to this system:

$$a = \frac{16}{17}, \qquad b = \frac{12}{17}, \qquad c = -\frac{16}{17}, \qquad d = \frac{4}{17}.$$

We have found that,

$$H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\big(s^2+s+\frac{5}{4}\big)} + \frac{(-4s+1)}{(s^2+1)}\Big].$$

Complete the square in the denominator,

$$s^2 + s + \frac{5}{4} = \Big[s^2 + 2\Big(\frac{1}{2}\Big)s + \frac{1}{4}\Big] - \frac{1}{4} + \frac{5}{4} = \Big(s + \frac{1}{2}\Big)^2 + 1.$$

$$H(s) = \frac{4}{17}\Big[\frac{(4s+3)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} + \frac{(-4s+1)}{(s^2+1)}\Big].$$

Rewrite the polynomial in the numerator,

$$(4s+3) = 4\Big(s + \frac{1}{2} - \frac{1}{2}\Big) + 3 = 4\Big(s + \frac{1}{2}\Big) + 1,$$

hence we get

$$H(s) = \frac{4}{17}\Big[4\,\frac{\big(s+\frac{1}{2}\big)}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} + \frac{1}{\big[\big(s+\frac{1}{2}\big)^2+1\big]} - 4\,\frac{s}{(s^2+1)} + \frac{1}{(s^2+1)}\Big].$$

Use the Laplace Transform Table in 2 to get $H(s)$ equal to

$$H(s) = \frac{4}{17}\Big[4\,\mathcal{L}\big[e^{-t/2}\cos(t)\big] + \mathcal{L}\big[e^{-t/2}\sin(t)\big] - 4\,\mathcal{L}[\cos(t)] + \mathcal{L}[\sin(t)]\Big],$$

equivalently

$$H(s) = \mathcal{L}\Big[\frac{4}{17}\Big(4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big)\Big].$$

Denote

$$h(t) = \frac{4}{17}\Big[4e^{-t/2}\cos(t) + e^{-t/2}\sin(t) - 4\cos(t) + \sin(t)\Big] \quad \Rightarrow \quad H(s) = \mathcal{L}[h(t)].$$

Recalling $\mathcal{L}[y(t)] = H(s) + e^{-\pi s}\,H(s)$, we obtain $\mathcal{L}[y(t)] = \mathcal{L}[h(t)] + e^{-\pi s}\,\mathcal{L}[h(t)]$, that is,

$$y(t) = h(t) + u(t-\pi)h(t-\pi).$$

◁

4.3.4. **Exercises.**

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4.3.2.- .

4.4. Generalized Sources

We introduce a generalized function, the Dirac Delta. We define the Dirac Delta as a limit $n \to \infty$ of a particular sequence of functions, $\{\delta_n\}$. We will see that this limit is a function on the domain $\mathbb{R} - \{0\}$, but it is not a function on $\mathbb{R}$. For that reason we call this limit a generalized function, the Dirac Delta generalized function.

We will show that each element in the sequence $\{\delta_n\}$ has a Laplace Transform, and this sequence of Laplace Transforms $\{\mathcal{L}[\delta_n]\}$ has a limit as $n \to \infty$. This limit of Laplace Transforms is how we define the Laplace Transform of the Dirac Delta.

We will solve differential equations having the Dirac Delta generalized function as source. Such differential equations appear often when one describes physical systems with impulsive forces, that is forces acting on a very short time but transfering a finite momentum to the system. Dirac's Delta is tailored to model impulsive forces.

4.4.1. Sequence of Functions and the Dirac Delta.

A sequence of functions is a sequence whose elements are functions. If each element in the sequence is a continuous function, we say that this is a sequence of continuous functions. It is not difficult to see that the limit of a sequence of continuous functions may be a continuous function. All the limits in this section are taken for a fixed value of the function independent variable. For example,

$$\Big\{ f_n(t) = \sin\Big(\Big(1 + \frac{1}{n}\Big)t\Big)\Big\} \to \sin(t) \quad \text{as} \quad n \to \infty,$$

where the limit is computed for each fixed value of t. However, not every sequence of continuous functions has a continuous function as a limit.

As an example, consider now the following sequence, $\{u_n\}$, for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases} \tag{4.4.1}$$

This is a sequence of continuous functions whose limit is a discontinuous function. From the few graphs in Fig. 22 we can see that the limit $n \to \infty$ of the sequence above is a step function, indeed, $\lim_{n\to\infty} u_n(t) = \tilde{u}(t)$, where

$$\tilde{u}(t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ 1 & \text{for} \quad t > 0. \end{cases}$$

We used a tilde in the name $\tilde{u}$ because this step function is not the same we defined in the previous section. The step u in § 4.3 satisfied $u(0) = 1$.

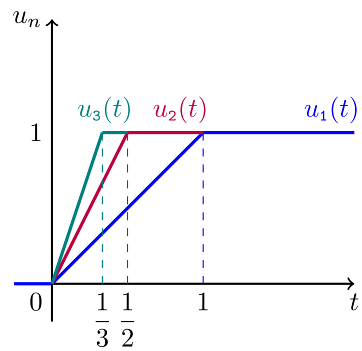


Figure 22. A few functions in the sequence $\{u_n\}$.

Exercise: Find a sequence $\{u_n\}$ so that its limit is the step function u defined in § 4.3.

Although every function in the sequence $\{u_n\}$ is continuous, the limit $\tilde{u}$ is a discontinuous function. It is not difficult to see that one can construct sequences of continuous functions having no limit at all. A similar situation happens when one considers sequences of piecewise discontinuous functions. In this case the limit could be a continuous function, a piecewise discontinuous function, or not a function at all.

We now introduce a particular sequence of piecewise discontinuous functions with domain $\mathbb{R}$ such that the limit as $n \to \infty$ does not exist for all values of the independent variable t.

The limit of the sequence is not a function with domain $\mathbb{R}$. In this case, the limit is a new type of object that we will call Dirac's Delta generalized function. Dirac's Delta is the limit of a sequence of particular bump functions.

Definition 4.4.1. *The* ***Dirac Delta*** *generalized function is the limit*

$$\delta(t) = \lim_{n\to\infty} \delta_n(t),$$

for every fixed $t \in \mathbb{R}$ of the sequence functions $\{\delta_n\}_{n=1}^{\infty}$,

$$\delta_n(t) = n\left[u(t) - u\left(t - \frac{1}{n}\right)\right], \tag{4.4.2}$$

The sequence of bump functions introduced above can be rewritten as follows,

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t < \dfrac{1}{n} \\ 0, & t \geqslant \dfrac{1}{n}. \end{cases}$$

We then obtain the equivalent expression,

$$\delta(t) = \begin{cases} 0 & \text{for} \quad t \neq 0, \\ \infty & \text{for} \quad t = 0. \end{cases}$$

Remark: It can be shown that there exist infinitely many sequences $\{\tilde{\delta}_n\}$ such that their limit as $n \to \infty$ is Dirac's Delta. For example, another sequence is

$$\tilde{\delta}_n(t) = n\left[u\left(t + \frac{1}{2n}\right) - u\left(t - \frac{1}{2n}\right)\right]$$
$$= \begin{cases} 0, & t < -\dfrac{1}{2n} \\ n, & -\dfrac{1}{2n} \leqslant t \leqslant \dfrac{1}{2n} \\ 0, & t > \dfrac{1}{2n}. \end{cases}$$

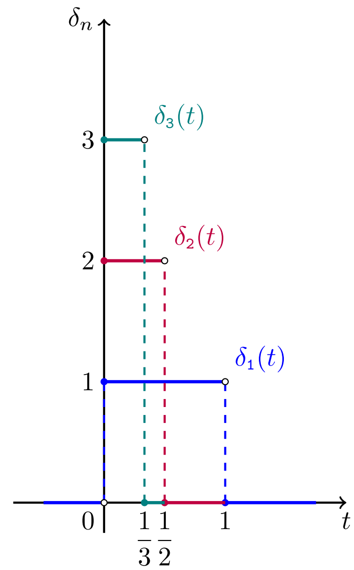


FIGURE 23. A few functions in the sequence $\{\delta_n\}$.

We see that the Dirac delta generalized function is a function on the domain $\mathbb{R} - \{0\}$. Actually it is the zero function on that domain. Dirac's Delta is not defined at $t = 0$, since the limit diverges at that point. One thing we can do now is to shift each element in the sequence by a real number c, and define

$$\delta(t - c) = \lim_{n\to\infty} \delta_n(t - c), \qquad c \in \mathbb{R}.$$

This shifted Dirac's Delta is identically zero on $\mathbb{R} - \{c\}$ and diverges at $t = c$. If we shift the graphs given in Fig. 23 by any real number c, one can see that

$$\int_c^{c+1} \delta_n(t - c)\, dt = 1$$

for every $n \geqslant 1$. Therefore, the sequence of integrals is the constant sequence, $\{1, 1, \cdots\}$, which has a trivial limit, 1, as $n \to \infty$. This says that the divergence at $t = c$ of the sequence $\{\delta_n\}$ is of a very particular type. The area below the graph of the sequence elements is always the same. We can say that this property of the sequence provides the main defining property of the Dirac Delta generalized function.

Using a limit procedure one can generalize several operations from a sequence to its limit. For example, translations, linear combinations, and multiplications of a function by a generalized function, integration and Laplace Transforms.

Definition 4.4.2. *We introduce the following operations on the Dirac Delta:*

$$f(t)\,\delta(t-c) + g(t)\,\delta(t-c) = \lim_{n\to\infty} \big[f(t)\,\delta_n(t-c) + g(t)\,\delta_n(t-c)\big],$$

$$\int_a^b \delta(t-c)\,dt = \lim_{n\to\infty} \int_a^b \delta_n(t-c)\,dt,$$

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)].$$

Remark: The notation in the definitions above could be misleading. In the left hand sides above we use the same notation as we use on functions, although Dirac's Delta is not a function on $\mathbb{R}$. Take the integral, for example. When we integrate a function f, the integration symbol means "take a limit of Riemann sums", that is,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\,\Delta x, \qquad x_i = a + i\,\Delta x, \qquad \Delta x = \frac{b-a}{n}.$$

However, when f is a generalized function in the sense of a limit of a sequence of functions $\{f_n\}$, then by the integration symbol we mean to compute a different limit,

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \int_a^b f_n(t)\,dt.$$

We use the same symbol, the integration, to mean two different things, depending whether we integrate a function or a generalized function. This remark also holds for all the operations we introduce on generalized functions, specially the Laplace Transform, that will be often used in the rest of this section.

4.4.2. **Computations with the Dirac Delta.** Once we have the definitions of operations involving the Dirac delta, we can actually compute these limits. The following statement summarizes few interesting results. The first formula below says that the infinity we found in the definition of Dirac's delta is of a very particular type; that infinity is such that Dirac's delta is integrable, in the sense defined above, with integral equal one.

Theorem 4.4.3. *For every $c \in \mathbb{R}$ and $\epsilon > 0$ holds,* $\displaystyle\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = 1.$

Proof of Theorem 4.4.3: The integral of a Dirac's delta generalized function is computed as a limit of integrals,

$$\begin{aligned}
\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt &= \lim_{n\to\infty} \int_{c-\epsilon}^{c+\epsilon} \delta_n(t-c)\,dt \\
&= \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\,dt, \qquad \frac{1}{n} < \epsilon, \\
&= \lim_{n\to\infty} n\Big(c + \frac{1}{n} - c\Big) \\
&= \lim_{n\to\infty} 1 \\
&= 1.
\end{aligned}$$

This establishes the Theorem. □

Theorem 4.4.4. *If f is continuous on (a,b) and $c \in (a,b)$, then* $\displaystyle\int_a^b f(t)\,\delta(t-c)\,dt = f(c)$.

Proof of Theorem 4.4.4: We again compute the integral of a Dirac's delta as a limit of a sequence of integrals,

$$\begin{aligned}\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty}\int_a^b \delta_n(t-c)\,f(t)\,dt\\ &= \lim_{n\to\infty}\int_a^b n\Big[u(t-c)-u\big(t-c-\frac{1}{n}\big)\Big]\,f(t)\,dt\\ &= \lim_{n\to\infty}\int_c^{c+\frac{1}{n}} n\,f(t)\,dt, \qquad \frac{1}{n}<(b-c),\end{aligned}$$

where in the last line we used that $c \in [a,b]$. If we denote by F any primitive of f, that is, $F' = f$, then we can write,

$$\begin{aligned}\int_a^b \delta(t-c)\,f(t)\,dt &= \lim_{n\to\infty} n\big[F\big(c+\frac{1}{n}\big)-F(c)\big]\\ &= \lim_{n\to\infty}\frac{1}{\big(\frac{1}{n}\big)}\big[F\big(c+\frac{1}{n}\big)-F(c)\big]\\ &= F'(c)\\ &= f(c).\end{aligned}$$

This establishes the Theorem. □

Theorem 4.4.5. *For all $s \in \mathbb{R}$ holds* $\mathcal{L}[\delta(t-c)] = \begin{cases} e^{-cs} & \text{for} \quad c \geqslant 0,\\ 0 & \text{for} \quad c<0.\end{cases}$

Proof of Theorem 4.4.5: The Laplace Transform of a Dirac's delta is computed as a limit of Laplace Transforms,

$$\begin{aligned}\mathcal{L}[\delta(t-c)] &= \lim_{n\to\infty}\mathcal{L}[\delta_n(t-c)]\\ &= \lim_{n\to\infty}\mathcal{L}\Big[n\big[u(t-c)-u\big(t-c-\frac{1}{n}\big)\big]\Big]\\ &= \lim_{n\to\infty}\int_0^\infty n\big[u(t-c)-u\big(t-c-\frac{1}{n}\big)\big]\,e^{-st}\,dt.\end{aligned}$$

The case $c<0$ is simple. For $\dfrac{1}{n}<|c|$ holds

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_0^\infty 0\,dt \quad\Rightarrow\quad \mathcal{L}[\delta(t-c)]=0, \qquad \text{for } s\in\mathbb{R}, \quad c<0.$$

Consider now the case $c \geqslant 0$. We then have,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_c^{c+\frac{1}{n}} n\,e^{-st}\,dt.$$

For $s=0$ we get

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_c^{c+\frac{1}{n}} n\,dt = 1 \quad\Rightarrow\quad \mathcal{L}[\delta(t-c)]=1 \quad \text{for } s=0, \quad c\geqslant 0.$$

In the case that $s \neq 0$ we get,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\int_c^{c+\frac{1}{n}} n\,e^{-st}\,dt = \lim_{n\to\infty} -\frac{n}{s}\big(e^{-cs}-e^{-(c+\frac{1}{n})s}\big) = e^{-cs}\lim_{n\to\infty}\frac{(1-e^{-\frac{s}{n}})}{\Big(\dfrac{s}{n}\Big)}.$$

The limit on the last line above is a singular limit of the form $\frac{0}{0}$, so we can use the l'Hôpital rule to compute it, that is,

$$\lim_{n\to\infty} \frac{(1-e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2}\, e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We then obtain,

$$\mathcal{L}[\delta(t-c)] = e^{-cs} \quad \text{for } s \neq 0, \quad c \geqslant 0.$$

This establishes the Theorem. □

4.4.3. **Applications of the Dirac Delta.** Dirac's Delta generalized functions describe *impulsive forces* in mechanical systems, such as the force done by a stick hitting a marble. An impulsive force acts on a very short time and transmits a finite momentum to the system.

Suppose we have a point particle with constant mass m. And to simplify the problem as much as we can, let us assume the particle can move along only one space direction, say x. If a force F acts on the particle, Newton's second law of motion says that

$$ma = F \quad \Leftrightarrow \quad mx''(t) = F(t, x(t)),$$

where the function values $x(t)$ are the particle position as function of time, $a(t) = x''(t)$ are the particle acceleration values, and we will denote $v(t) = x'(t)$ the particle velocity values. We saw in § 1.1 that Newton's second law of motion is a second order differential equation for the position function x. Now it is more convenient to use the *particle momentum*, $p = mv$, to write the Newton's equation,

$$mx'' = mv' = (mv)' = F \quad \Rightarrow \quad p' = F.$$

Writing Newton's equation in this form it is simpler to see that forces change the particle momentum. Integrating in time on an interval $[t_1, t_2]$ we get

$$\Delta p = p(t_2) - p(t_1) = \int_{t_1}^{t_2} F(t, x(t))\, dt.$$

Suppose that an impulsive force is acting on a particle at t_0 transmitting a finite momentum, say p_0. This is where the Dirac Delta is uselful for, because we can write the force as

$$F(t) = p_0\, \delta(t - t_0),$$

then $F = 0$ on $\mathbb{R} - \{t_0\}$ and the momentum transferred to the particle by the force is

$$\Delta p = \int_{t_0-\Delta t}^{t_0+\Delta t} p_0\, \delta(t-t_0)\, dt = p_0.$$

The momentum tranferred is $\Delta p = p_0$, but the force is identically zero on $\mathbb{R} - \{t_0\}$. We have transferred a finite momentum to the particle by an interaction at a single time t_0.

4.4.4. **The Impulse Response Function.** We now want to solve differential equations with the Dirac Delta as a source. But a particular type of solutions will be important later on, those solutions to initial value problems with the Dirac Delta generalized function as a source and zero initial conditions. We give these solutions a particular name.

Definition 4.4.6. *The **impulse response function** at the point $c \geqslant 0$ of the linear operator $L(y) = y'' + a_1\, y' + a_0\, y$, with a_1, a_0 constants, is the solution y_δ, in the sense of Laplace Transforms, of the initial value problem*

$$L(y_\delta) = \delta(t-c), \qquad y_\delta(0) = 0, \qquad y'_\delta(0) = 0, \qquad c \geqslant 0.$$

Remark: Impulse response functions are also called *fundamental solutions*.

EXAMPLE 4.4.1: Find the impulse response function at $t = 0$ of the linear operator

$$L(y) = y'' + 2y' + 2y.$$

SOLUTION: We need to find the solution y_δ of the initial value problem

$$y_\delta'' + 2y_\delta' + 2y_\delta = \delta(t), \qquad y_\delta(0) = 0, \qquad y_\delta'(0) = 0.$$

Since the souce is a Dirac Delta, we have to use the Laplace Transform to solve this problem. So we compute the Laplace Transform on both sides of the differential equation,

$$\mathcal{L}[y_\delta''] + 2\,\mathcal{L}[y_\delta'] + 2\,\mathcal{L}[y_\delta] = \mathcal{L}[\delta(t)] = 1 \quad \Rightarrow \quad (s^2 + 2s + 2)\,\mathcal{L}[y_\delta] = 1,$$

where we have introduced the initial conditions on the last equation above. So we obtain

$$\mathcal{L}[y_\delta] = \frac{1}{(s^2 + 2s + 2)}.$$

The denominator in the equation above has complex valued roots, since

$$s_\pm = \frac{1}{2}\big[-2 \pm \sqrt{4 - 8}\big],$$

therefore, we complete squares $s^2 + 2s + 2 = (s+1)^2 + 1$. We need to solve the equation

$$\mathcal{L}[y_\delta] = \frac{1}{\big[(s+1)^2 + 1\big]} = \mathcal{L}[e^{-t}\sin(t)] \quad \Rightarrow \quad y_\delta(t) = e^{-t}\sin(t).$$

◁

EXAMPLE 4.4.2: Find the impulse response function at $t = c \geqslant 0$ of the linear operator

$$L(y) = y'' + 2\,y' + 2\,y.$$

SOLUTION: We need to find the solution y_δ of the initial value problem

$$y_\delta'' + 2\,y_\delta' + 2\,y_\delta = \delta(t - c), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

We have to use the Laplace Transform to solve this problem because the source is a Dirac's Delta generalized function. So, compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y_\delta''] + 2\,\mathcal{L}[y_\delta'] + 2\,\mathcal{L}[y_\delta] = \mathcal{L}[\delta(t - c)].$$

Since the initial conditions are all zero and $c \geqslant 0$, we get

$$(s^2 + 2s + 2)\,\mathcal{L}[y_\delta] = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s^2 + 2s + 2)}.$$

Find the roots of the denominator,

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\,\big[-2 \pm \sqrt{4 - 8}\big]$$

The denominator has complex roots. Then, it is convenient to complete the square in the denominator,

$$s^2 + 2s + 2 = \Big[s^2 + 2\Big(\frac{2}{2}\Big)\,s + 1\Big] - 1 + 2 = (s+1)^2 + 1.$$

Therefore, we obtain the expression,

$$\mathcal{L}[y_\delta] = \frac{e^{-cs}}{(s+1)^2 + 1}.$$

Recall that $\mathcal{L}[\sin(t)] = \dfrac{1}{s^2 + 1}$, and $\mathcal{L}[f](s - c) = \mathcal{L}[e^{ct}\,f(t)]$. Then,

$$\frac{1}{(s+1)^2 + 1} = \mathcal{L}[e^{-t}\,\sin(t)] \quad \Rightarrow \quad \mathcal{L}[y_\delta] = e^{-cs}\,\mathcal{L}[e^{-t}\,\sin(t)].$$

Since for $c \geqslant 0$ holds $e^{-cs}\,\mathcal{L}[f](s) = \mathcal{L}[u(t-c)\,f(t-c)]$, we conclude that

$$y_\delta(t) = u(t-c)\,e^{-(t-c)}\,\sin(t-c).$$

◁

Example 4.4.3: Find the solution y to the initial value problem

$$y'' - y = -20\,\delta(t-3), \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: The source is a generalized function, so we need to solve this problem using the Lapace Transform. So we compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y''] - \mathcal{L}[y] = -20\,\mathcal{L}[\delta(t-3)] \quad \Rightarrow \quad (s^2-1)\,\mathcal{L}[y] - s = -20\,e^{-3s},$$

where in the second equation we have already introduced the initial conditions. We arrive to the equation

$$\mathcal{L}[y] = \frac{s}{(s^2-1)} - 20\,e^{-3s}\,\frac{1}{(s^2-1)} = \mathcal{L}[\cosh(t)] - 20\,\mathcal{L}[u(t-3)\,\sinh(t-3)],$$

which leads to the solution

$$y(t) = \cosh(t) - 20\,u(t-3)\,\sinh(t-3).$$

◁

Example 4.4.4: Find the solution to the initial value problem

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: We again Laplace Transform both sides of the differential equation,

$$\mathcal{L}[y''] + 4\,\mathcal{L}[y] = \mathcal{L}[\delta(t-\pi)] - \mathcal{L}[\delta(t-2\pi)] \quad \Rightarrow \quad (s^2+4)\,\mathcal{L}[y] = e^{-\pi s} - e^{-2\pi s},$$

where in the second equation above we have introduced the initial conditions. Then,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{e^{-\pi s}}{(s^2+4)} - \frac{e^{-2\pi s}}{(s^2+4)} \\
&= \frac{e^{-\pi s}}{2}\,\frac{2}{(s^2+4)} - \frac{e^{-2\pi s}}{2}\,\frac{2}{(s^2+4)} \\
&= \frac{1}{2}\mathcal{L}\Big[u(t-\pi)\,\sin\big[2(t-\pi)\big]\Big] - \frac{1}{2}\mathcal{L}\Big[u(t-2\pi)\,\sin\big[2(t-2\pi)\big]\Big].
\end{aligned}$$

The last equation can be rewritten as follows,

$$y(t) = \frac{1}{2}\,u(t-\pi)\,\sin\big[2(t-\pi)\big] - \frac{1}{2}\,u(t-2\pi)\,\sin\big[2(t-2\pi)\big],$$

which leads to the conclusion that

$$y(t) = \frac{1}{2}\,\big[u(t-\pi) - u(t-2\pi)\big]\,\sin(2t).$$

◁

4.4.5. **Comments on Generalized Sources.** We have used the Laplace Transform to solve differential equations with the Dirac Delta as a source function. It may be convenient to understand a bit more clearly what we have done, since the Dirac Delta is not an ordinary function but a generalized function defined by a limit. Consider the following example.

EXAMPLE 4.4.5: Find the impulse response function at $t = c > 0$ of the linear operator

$$L(y) = y'.$$

SOLUTION: We need to solve the initial value problem

$$y'(t) = \delta(t-c), \qquad y(0) = 0.$$

In other words, we need to find a primitive of the Dirac Delta. However, Dirac's Delta is not even a function. Anyway, let us compute the Laplace Transform of the equation, as we did in the previous examples,

$$\mathcal{L}[y'(t)] = \mathcal{L}[\delta(t-c)] \quad \Rightarrow \quad s\,\mathcal{L}[y(t)] - y(0) = e^{-cs} \quad \Rightarrow \quad \mathcal{L}[y(t)] = \frac{e^{-cs}}{s}.$$

But we know that

$$\frac{e^{-cs}}{s} = \mathcal{L}[u(t-c)] \quad \Rightarrow \quad \mathcal{L}[y(t)] = \mathcal{L}[u(t-c)] \quad \Rightarrow \quad y(t) = u(t-c).$$

◁

Looking at the differential equation $y'(t) = \delta(t-c)$ and at the solution $y(t) = u(t-c)$ one could like to write them together as

$$u'(t-c) = \delta(t-c). \tag{4.4.3}$$

But this is not correct, because the step function is a discontinuous function at $t = c$, hence not differentiable. What we have done is something different. We have found a sequence of functions u_n with the properties,

$$\lim_{n\to\infty} u_n(t-c) = u(t-c), \qquad \lim_{n\to\infty} u_n'(t-c) = \delta(t-c),$$

and we have called $y(t) = u(t-c)$. This is what we actually do when we solve a differential equation with a source defined as a limit of a sequence of functions, such as the Dirac Delta. The Laplace Transform Method used on differential equations with generalized sources allows us to solve these equations without the need to write any sequence, which are hidden in the definitions of the Laplace Transform of generalized functions. Let us solve the problem in the Example 4.4.5 one more time, but this time let us show where all the sequences actually are.

EXAMPLE 4.4.6: Find the solution to the initial value problem

$$y'(t) = \delta(t-c), \qquad y(0) = 0, \qquad c > 0, \tag{4.4.4}$$

SOLUTION: Recall that the Dirac Delta is defined as a limit of a sequence of bump functions,

$$\delta(t-c) = \lim_{n\to\infty} \delta_n(t-c), \qquad \delta_n(t-c) = n\Big[u(t-c) - u\Big(t-c-\frac{1}{n}\Big)\Big], \qquad n = 1, 2, \cdots.$$

The problem we are actually solving involves a sequence and a limit,

$$y'(t) = \lim_{n\to\infty} \delta_n(t-c), \qquad y(0) = 0.$$

We start computing the Laplace Transform of the differential equation,

$$\mathcal{L}[y'(t)] = \mathcal{L}[\lim_{n\to\infty} \delta_n(t-c)].$$

We have defined the Laplace Transform of the limit as the limit of the Laplace Transforms,

$$\mathcal{L}[y'(t)] = \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)].$$

If the solution is at least piecewise differentiable, we can use the property

$$\mathcal{L}[y'(t)] = s\,\mathcal{L}[y(t)] - y(0).$$

Assuming that property, and the initial condition $y(0) = 0$, we get

$$\mathcal{L}[y(t)] = \frac{1}{s} \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] \quad \Rightarrow \quad \mathcal{L}[y(t)] = \lim_{n\to\infty} \frac{\mathcal{L}[\delta_n(t-c)]}{s}.$$

Introduce now the function $y_n(t) = u_n(t-c)$, given in Eq. (4.4.1), which for each n is the only continuous, piecewise differentiable, solution of the initial value problem

$$y_n'(t) = \delta_n(t-c), \qquad y_n(0) = 0.$$

It is not hard to see that this function u_n satisfies

$$\mathcal{L}[u_n(t)] = \frac{\mathcal{L}[\delta_n(t-c)]}{s}.$$

Therefore, using this formula back in the equation for y we get,

$$\mathcal{L}[y(t)] = \lim_{n\to\infty} \mathcal{L}[u_n(t)].$$

For continuous functions we can interchange the Laplace Transform and the limit,

$$\mathcal{L}[y(t)] = \mathcal{L}[\lim_{n\to\infty} u_n(t)].$$

So we get the result,

$$y(t) = \lim_{n\to\infty} u_n(t) \quad \Rightarrow \quad y(t) = u(t-c).$$

We see above that we have found something more than just $y(t) = u(t-c)$. We have found

$$y(t) = \lim_{n\to\infty} u_n(t-c),$$

where the sequence elements u_n are continuous functions with $u_n(0) = 0$ and

$$\lim_{n\to\infty} u_n(t-c) = u(t-c), \qquad \lim_{n\to\infty} u_n'(t-c) = \delta(t-c),$$

Finally, derivatives and limits cannot be interchanged for u_n,

$$\lim_{n\to\infty} \big[u_n'(t-c)\big] \neq \big[\lim_{n\to\infty} u_n(t-c)\big]'$$

so it makes no sense to talk about y'. ◁

When the Dirac Delta is defined by a sequence of functions, as we did in this section, the calculation needed to find impulse response functions must involve sequence of functions and limits. The Laplace Transform Method used on generalized functions allows us to hide all the sequences and limits. This is true not only for the derivative operator $L(y) = y'$ but for any second order differential operator with constant coefficients.

Definition 4.4.7. *A **solution** of the initial value problem with a Dirac's Delta source*

$$y'' + a_1\, y' + a_0\, y = \delta(t-c), \qquad y(0) = y_0, \qquad y'(0) = y_1, \tag{4.4.5}$$

where a_1, a_0, y_0, y_1, and $c \in \mathbb{R}$, are given constants, is a function

$$y(t) = \lim_{n\to\infty} y_n(t),$$

where the functions y_n, with $n \geqslant 1$, are the unique solutions to the initial value problems

$$y_n'' + a_1\, y_n' + a_0\, y_n = \delta_n(t-c), \qquad y_n(0) = y_0, \qquad y_n'(0) = y_1, \tag{4.4.6}$$

and the source δ_n satisfy $\lim_{n\to\infty} \delta_n(t-c) = \delta(t-c)$.

The definition above makes clear what do we mean by a solution to an initial value problem having a generalized function as source, when the generalized function is defined as the limit of a sequence of functions. The following result says that the Laplace Transform Method used with generalized functions hides all the sequence computations.

Theorem 4.4.8. *The function y is solution of the initial value problem*

$$y'' + a_1\, y' + a_0\, y = \delta(t-c), \qquad y(0) = y_0, \qquad y'(0) = y_1, \qquad c \geqslant 0,$$

iff its Laplace Transform satisfies the equation

$$\big(s^2\, \mathcal{L}[y] - sy_0 - y_1\big) + a_1\, \big(s\, \mathcal{L}[y] - y_0\big) - a_0\, \mathcal{L}[y] = e^{-cs}.$$

This Theorem tells us that to find the solution y to an initial value problem when the source is a Dirac's Delta we have to apply the Laplace Transform to the equation and perform the same calculations as if the Dirac Delta were a function. This is the calculation we did when we computed the impulse response functions.

Proof of Theorem 4.4.8: Compute the Laplace Transform on Eq. (4.4.6),

$$\mathcal{L}[y_n''] + a_1\, \mathcal{L}[y_n'] + a_0\, \mathcal{L}[y_n] = \mathcal{L}[\delta_n(t-c)].$$

Recall the relations between the Laplace Transform and derivatives and use the initial conditions,

$$\mathcal{L}[y_n''] = s^2\, \mathcal{L}[y_n] - sy_0 - y_1, \qquad \mathcal{L}[y'] = s\, \mathcal{L}[y_n] - y_0,$$

and use these relation in the differential equation,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y_n] - sy_0 - y_1 - a_1 y_0 = \mathcal{L}[\delta_n(t-c)],$$

Since δ_n satisfies that $\lim_{n\to\infty} \delta_n(t-c) = \delta(t-c)$, an argument like the one in the proof of Theorem 4.4.5 says that for $c \geqslant 0$ holds

$$\mathcal{L}[\delta_n(t-c)] = \mathcal{L}[\delta(t-c)] \quad \Rightarrow \quad \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] = e^{-cs}.$$

Then

$$(s^2 + a_1 s + a_0)\, \lim_{n\to\infty} \mathcal{L}[y_n] - sy_0 - y_1 - a_1 y_0 = e^{-cs}.$$

Interchanging limits and Laplace Transforms we get

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] - sy_0 - y_1 - a_1 y_0 = e^{-cs},$$

which is equivalent to

$$\big(s^2\, \mathcal{L}[y] - sy_0 - y_1\big) + a_1\, \big(s\, \mathcal{L}[y] - y_0\big) - a_0\, \mathcal{L}[y] = e^{-cs}.$$

This establishes the Theorem. □

4.4.6. **Exercises.**

4.4.1.- .

4.4.2.- .

4.5. Convolutions and Solutions

Solutions of initial value problems for linear nonhomogeneous differential equations can be decomposed in a nice way. The part of the solution coming from the initial data can be separated from the part of the solution coming from the nonhomogeneous source function. Furthermore, the latter is a kind of product of two functions, the source function itself and the impulse response function from the differential operator. This kind of product of two functions is the subject of this section. This kind of product is what we call the convolution of two functions.

4.5.1. Definition and Properties.

One can say that the convolution is a generalization of the pointwise product of two functions. In a convolution one multiplies the two functions evaluated at different points and then integrates the result. Here is a precise definition.

Definition 4.5.1. *The **convolution** of functions f and g is a function $f * g$ given by*

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{4.5.1}$$

Remark: The convolution is defined for functions f and g such that the integral in (4.5.1) is defined. For example for f and g piecewise continuous functions, or one of them continuous and the other a Dirac's Delta generalized function.

Example 4.5.1: Find $f * g$ the convolution of the functions $f(t) = e^{-t}$ and $g(t) = \sin(t)$.

Solution: The definition of convolution is,

$$(f * g)(t) = \int_0^t e^{-\tau}\sin(t-\tau)\,d\tau.$$

This integral is not difficult to compute. Integrate by parts twice,

$$\int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau = \Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t - \int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau,$$

that is,

$$2\int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau = \Big[e^{-\tau}\cos(t-\tau)\Big]\Big|_0^t - \Big[e^{-\tau}\sin(t-\tau)\Big]\Big|_0^t = e^{-t} - \cos(t) - 0 + \sin(t).$$

We then conclude that

$$(f * g)(t) = \frac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big]. \tag{4.5.2}$$

◁

A few properties of the convolution operation are summarized in the Theorem below. But we save the most important property for the next subsection.

Theorem 4.5.2 (Properties). *For every piecewise continuous functions f, g, and h, hold:*

(i) Commutativity: $f * g = g * f$;

(ii) Associativity: $f * (g * h) = (f * g) * h$;

(iii) Distributivity: $f * (g + h) = f * g + f * h$;

(iv) Neutral element: $f * 0 = 0$;

(v) Identity element: $f * \delta = f$.

Proof of Theorem 4.5.2: We only prove properties (i) and (v), the rest are left as an exercise and they are not so hard to obtain from the definition of convolution. The first property can be obtained by a change of the integration variable as follows,

$$\begin{aligned}(f*g)(t) &= \int_0^t f(\tau)\,g(t-\tau)\,d\tau, \qquad \hat\tau = t-\tau, \qquad d\hat\tau = -d\tau,\\ &= \int_t^0 f(t-\hat\tau)\,g(\hat\tau)(-1)\,d\hat\tau\\ &= \int_0^t g(\hat\tau)\,f(t-\hat\tau)\,d\hat\tau \quad\Rightarrow\quad (f*g)(t) = (g*f)(t).\end{aligned}$$

We now move to property (v), which is essentially a property of the Dirac Delta,

$$(f*\delta)(t) = \int_0^t f(\tau)\,\delta(t-\tau)\,d\tau = f(t).$$

This establishes the Theorem. □

4.5.2. **The Laplace Transform.** The Laplace Transform of a convolution of two functions is the pointwise product of their corresponding Laplace Transforms. This result will be a key part in the solution decomposition result we show at the end of the section.

Theorem 4.5.3 (Laplace Transform). *If the functions f and g have Laplace Transforms $\mathcal{L}[f]$ and $\mathcal{L}[g]$, including the case where one of them is a Dirac's Delta, then*

$$\mathcal{L}[f*g] = \mathcal{L}[f]\,\mathcal{L}[g]. \tag{4.5.3}$$

Remark: It is not an accident that the convolution of two functions satisfies Eq. (4.5.3). The definition of convolution is chosen so that it has this property. One can see that this is the case by looking at the proof of Theorem 4.5.3. One starts with the expression $\mathcal{L}[f]\,\mathcal{L}[g]$, then changes the order of integration, and one ends up with the Laplace Transform of some quantity. Because this quantity appears in that expression, is that it deserves a name. This is how the convolution operation was created.

Example 4.5.2: Compute the Laplace Transform of the function $u(t) = \displaystyle\int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau$.

Solution: The function u above is the convolution of the functions

$$f(t) = e^{-t}, \qquad g(t) = \sin(t),$$

that is, $u = f*g$. Therefore, Theorem 4.5.3 says that

$$\mathcal{L}[u] = \mathcal{L}[f*g] = \mathcal{L}[f]\,\mathcal{L}[g].$$

Since,

$$\mathcal{L}[f] = \mathcal{L}[e^{-t}] = \frac{1}{s+1}, \qquad \mathcal{L}[g] = \mathcal{L}[\sin(t)] = \frac{1}{s^2+1},$$

we then conclude that $\mathcal{L}[u] = \mathcal{L}[f*g]$ is given by

$$\mathcal{L}[f*g] = \frac{1}{(s+1)(s^2+1)}.$$

◁

Proof of Theorem 4.5.3: We start writing the right hand side of Eq. (4.5.1), the product $\mathcal{L}[f]\,\mathcal{L}[g]$. We write the two integrals coming from the individual Laplace Transforms and we rewrite them in an appropriate way.

$$\begin{aligned}\mathcal{L}[f]\,\mathcal{L}[g] &= \Big[\int_0^\infty e^{-st} f(t)\,dt\Big]\,\Big[\int_0^\infty e^{-s\tilde t} g(\tilde t)\,d\tilde t\Big]\\ &= \int_0^\infty e^{-s\tilde t} g(\tilde t)\Big(\int_0^\infty e^{-st} f(t)\,dt\Big)\,d\tilde t\\ &= \int_0^\infty g(\tilde t)\Big(\int_0^\infty e^{-s(t+\tilde t)} f(t)\,dt\Big)\,d\tilde t,\end{aligned}$$

where we only introduced the integral in t as a constant inside the integral in $\tilde t$. Introduce the change of variables in the inside integral $\tau = t + \tilde t$, hence $d\tau = dt$. Then, we get

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty g(\tilde t)\Big(\int_{\tilde t}^\infty e^{-s\tau} f(\tau - \tilde t)\,d\tau\Big)\,d\tilde t \tag{4.5.4}$$

$$= \int_0^\infty \int_{\tilde t}^\infty e^{-s\tau}\, g(\tilde t)\, f(\tau - \tilde t)\,d\tau\,d\tilde t. \tag{4.5.5}$$

Here is the key step. We must switch the order of integration. From Fig. 24 we see that changing the order of integration gives the following expression,

$$\mathcal{L}[f]\,\mathcal{L}[g] = \int_0^\infty \int_0^\tau e^{-s\tau}\, g(\tilde t)\, f(\tau - \tilde t)\,d\tilde t\,d\tau.$$

Then, is straightforward to check that

$$\begin{aligned}\mathcal{L}[f]\,\mathcal{L}[g] &= \int_0^\infty e^{-s\tau}\,\Big(\int_0^\tau g(\tilde t)\, f(\tau - \tilde t)\,d\tilde t\Big)\,d\tau\\ &= \int_0^\infty e^{-s\tau} (g * f)(\tau)\,dt\\ &= \mathcal{L}[g * f] \quad \Rightarrow \quad \mathcal{L}[f]\,\mathcal{L}[g] = \mathcal{L}[f * g].\end{aligned}$$

This establishes the Theorem. □

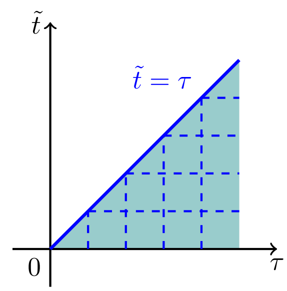


FIGURE 24. Domain of integration in (4.5.5).

EXAMPLE 4.5.3: Use the Laplace Transform to compute $u(t) = \displaystyle\int_0^t e^{-\tau}\,\sin(t-\tau)\,d\tau$.

SOLUTION: We know that $u = f * g$, with $f(t) = e^{-t}$ and $g(t) = \sin(t)$, and we have seen in Example 4.5.2 that

$$\mathcal{L}[u] = \mathcal{L}[f * g] = \frac{1}{(s+1)(s^2+1)}.$$

A partial fraction decomposition of the right hand side above implies that

$$\begin{aligned}\mathcal{L}[u] &= \frac{1}{2}\Big[\frac{1}{(s+1)} + \frac{(1-s)}{(s^2+1)}\Big]\\ &= \frac{1}{2}\Big[\frac{1}{(s+1)} + \frac{1}{(s^2+1)} - \frac{s}{(s^2+1)}\Big]\\ &= \frac{1}{2}\Big(\mathcal{L}[e^{-t}] + \mathcal{L}[\sin(t)] - \mathcal{L}[\cos(t)]\Big).\end{aligned}$$

This says that

$$u(t) = \frac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big].$$

We then conclude that

$$(f * g)(t) = \frac{1}{2}\big[e^{-t} + \sin(t) - \cos(t)\big],$$

which agrees with Eq. (4.5.2) in the first example above. ◁

4.5.3. **Solution Decomposition.** The Solution Decomposition Theorem is the main result of this section. Theorem 4.5.4 shows one way to write the solution to a general initial value problem for a linear second order differential equation with constant coefficients. The solution to such problem can always be divided in two terms. The first term contains information only about the initial data. The second term contains information only about the source function. This second term is a convolution of the source function itself and the impulse response function of the differential operator.

Theorem 4.5.4 (Solution Decomposition). *Given constants a_0, a_1, y_0, y_1 and a piecewise continuous function g, the solution y to the initial value problem*

$$y'' + a_1\, y' + a_0\, y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1, \tag{4.5.6}$$

can be decomposed as

$$y(t) = y_h(t) + (y_\delta * g)(t), \tag{4.5.7}$$

where y_h is the solution of the homogeneous initial value problem

$$y_h'' + a_1\, y_h' + a_0\, y_h = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1, \tag{4.5.8}$$

and y_δ is the impulse response solution, that is,

$$y_\delta'' + a_1\, y_\delta' + a_0\, y_\delta = \delta(t), \quad y_\delta(0) = 0, \quad y_\delta'(0) = 0.$$

Remark: The solution decomposition in Eq. (4.5.7) can be written in the equivalent way

$$y(t) = y_h(t) + \int_0^t y_\delta(\tau) g(t-\tau)\, d\tau.$$

Proof of Theorem4.5.4: Compute the Laplace Transform of the differential equation,

$$\mathcal{L}[y''] + a_1\, \mathcal{L}[y'] + a_0\, \mathcal{L}[y] = \mathcal{L}[g(t)].$$

Recalling the relations between Laplace Transforms and derivatives,

$$\mathcal{L}[y''] = s^2\, \mathcal{L}[y] - s y_0 - y_1, \qquad \mathcal{L}[y'] = s\, \mathcal{L}[y] - y_0.$$

we re-write the differential equation for y as an algebraic equation for $cL[y]$,

$$(s^2 + a_1 s + a_0)\, \mathcal{L}[y] - s y_0 - y_1 - a_1 y_0 = \mathcal{L}[g(t)].$$

As usual, it is simple to solve the algebraic equation for $cL[y]$,

$$\mathcal{L}[y] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)} + \frac{1}{(s^2 + a_1 s + a_0)}\, \mathcal{L}[g(t)].$$

Now, the function y_h is the solution of Eq. (4.5.8), that is,

$$\mathcal{L}[y_h] = \frac{(s + a_1) y_0 + y_1}{(s^2 + a_1 s + a_0)}.$$

And by the definition of the impulse response solution y_δ we have that

$$\mathcal{L}[y_\delta] = \frac{1}{(s^2 + a_1 s + a_0)}.$$

These last three equation imply,

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\, \mathcal{L}[g(t)].$$

This is the Laplace Transform version of Eq. (4.5.7). Inverting the Laplace Transform above,

$$y(t) = y_h(t) + \mathcal{L}^{-1}\big[\mathcal{L}[y_\delta]\, \mathcal{L}[g(t)]\big].$$

Using the result in Theorem 4.5.3 in the last term above we conclude that

$$y(t) = y_h(t) + (y_\delta * g)(t).$$

□

Example 4.5.4: Use the Solution Decomposition Theorem to express the solution of

$$y'' + 2\,y' + 2\,y = \sin(at), \quad y(0) = 1, \quad y'(0) = -1.$$

Solution: Compute the Laplace Transform of the differential equation above,

$$\mathcal{L}[y''] + 2\,\mathcal{L}[y'] + 2\,\mathcal{L}[y] = \mathcal{L}[\sin(at)],$$

and recall the relations between the Laplace Transform and derivatives,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - sy(0) - y'(0), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0).$$

Introduce the initial conditions in the equation above,

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,(1) - (-1), \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - 1,$$

and these two equation into the differential equation,

$$(s^2 + 2s + 2)\,\mathcal{L}[y] - s + 1 - 2 = \mathcal{L}[\sin(at)].$$

Reorder terms to get

$$\mathcal{L}[y] = \frac{(s+1)}{(s^2+2s+2)} + \frac{1}{(s^2+2s+2)}\,\mathcal{L}[\sin(at)].$$

Now, the function y_h is the solution of the homogeneous initial value problem with the same initial conditions as y, that is,

$$\mathcal{L}[y_h] = \frac{(s+1)}{(s^2+2s+2)} = \frac{(s+1)}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\cos(t)].$$

Now, the function y_δ is the impulse response solution for the differential equation in this Example, that is,

$$cL[y_\delta] = \frac{1}{(s^2+2s+2)} = \frac{1}{(s+1)^2+1} = \mathcal{L}[e^{-t}\,\sin(t)].$$

Put all this information together and denote $g(t) = \sin(at)$ to get

$$\mathcal{L}[y] = \mathcal{L}[y_h] + \mathcal{L}[y_\delta]\,\mathcal{L}[g(t)] \quad \Rightarrow \quad y(t) = y_h(t) + (y_\delta * g)(t),$$

More explicitly, we get

$$y(t) = e^{-t}\,\cos(t) + \int_0^t e^{-\tau}\sin(\tau)\,\sin[a(t-\tau)]\,d\tau.$$

◁

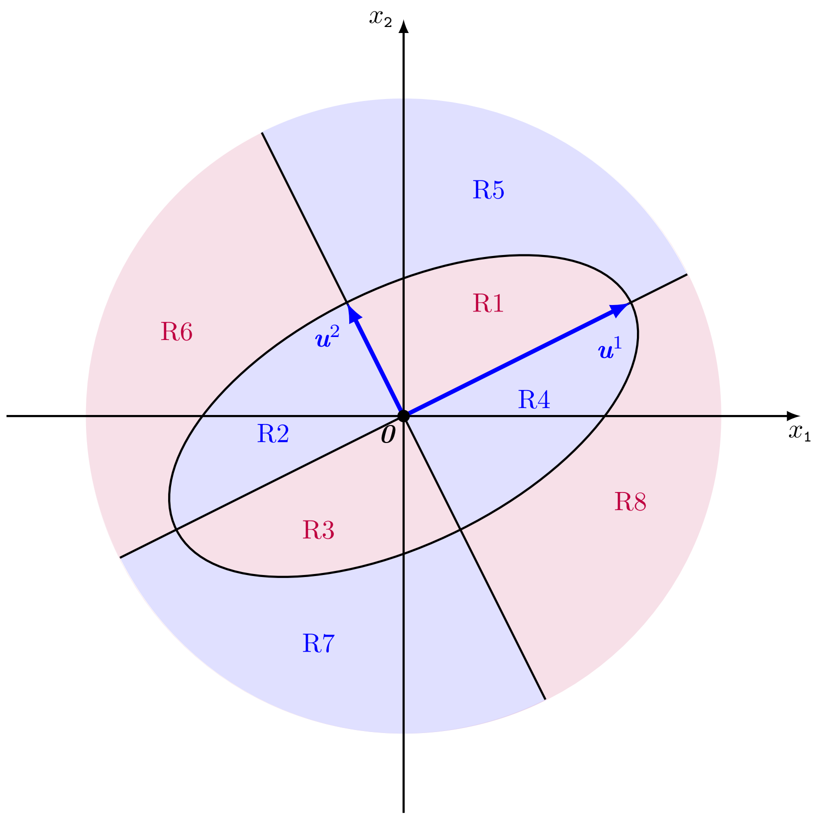
4.5.4. **Exercises.**

4.5.1.- .

4.5.2.- .

Chapter 5. Systems of Differential Equations

Newton's second law of motion for point particles is one of the first differential equations ever written. Even this early example of a differential equation consists not of a single equation but of a system of three equation on three unknowns. The unknown functions are the particle three coordinates in space as function of time. One important difficulty to solve a differential system is that the equations in a system are usually coupled. One cannot solve for one unknown function without knowing the other unknowns. In this chapter we study how to solve the system in the particular case that the equations can be uncoupled. We call such systems diagonalizable. Explicit formulas for the solutions can be written in this case. Later we generalize this idea to systems that cannot be uncoupled.



5.1. Linear Differential Systems

We introduce a linear differential system with variable coefficients. We present an initial value problem for such systems and we state that initial value problems always have a unique solution. The proof is based on a generalization of the Picard-Lindelöf iteration used in Section 1.6. We then introduce the concepts of fundamental solution, general solution, fundamental matrix, and Wronskian of solutions to a linear system. This section is a generalization of the ideas in § 2.1 from a single equation to a system of equations.

5.1.1. First Order Linear Systems.

A single differential equation on one unknown function is often not enough to describe certain physical problems. The description of a point particle moving in space under Newton's law of motion requires three functions of time, the space coordinates of the particle, to describe the motion together with three differential equations. To describe several proteins activating and deactivating each other inside a cell also requires as many unknown functions and equations as proteins in the system. In this Section we present a first step aimed to describe such physical systems. We start introducing a first order linear differential system.

Definition 5.1.1. *An $n \times n$* ***first order linear differential system*** *is the equation*

$$\boldsymbol{x}'(t) = A(t)\,\boldsymbol{x}(t) + \boldsymbol{g}(t), \tag{5.1.1}$$

where the $n \times n$ coefficient matrix A, the source n-vector $\boldsymbol{g}$, and the unknown n-vector $\boldsymbol{x}$ are given in components by

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \qquad \boldsymbol{g}(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

The system is called ***homogeneous*** *iff the source vector $\boldsymbol{g} = \boldsymbol{0}$. The system is called of* ***constant coefficients*** *iff the coefficient matrix A is constant.*

Remarks:

(a) The derivative of a a vector-valued function is defined as $\boldsymbol{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$.

(b) By the definition of the matrix-vector product, Eq. (5.1.1) can be written as

$$\begin{aligned} x_1'(t) &= a_{11}(t)\,x_1(t) + \cdots + a_{1n}(t)\,x_n(t) + g_1(t), \\ &\vdots \\ x_n'(t) &= a_{n1}(t)\,x_1(t) + \cdots + a_{nn}(t)\,x_n(t) + g_n(t). \end{aligned}$$

A *solution* of an $n \times n$ linear differential system is an n-vector-valued function $\boldsymbol{x}$, that is, a set of n functions $\{x_1, \cdots, x_n\}$, that satisfy every differential equation in the system. When we write down the equations we will usually write $\boldsymbol{x}$ instead of $\boldsymbol{x}(t)$.

Example 5.1.1: The case $n = 1$ is a single differential equation: Find a solution x_1 of

$$x_1' = a_{11}(t)\,x_1 + g_1(t).$$

This is a linear first order equation, and solutions can be found with the integrating factor method described in Section 1.2. ◁

EXAMPLE 5.1.2: The case $n = 2$ is a 2×2 linear system: Find functions x_1, x_2 solutions of

$$x_1' = a_{11}(t)\, x_1 + a_{12}(t)\, x_2 + g_1(t),$$
$$x_2' = a_{21}(t)\, x_1 + a_{22}(t)\, x_2 + g_2(t).$$

In this case the coefficient matrix A, the source vector $\boldsymbol{g}$, and the unknown vector $\boldsymbol{x}$ are,

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, \qquad \boldsymbol{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

◁

EXAMPLE 5.1.3: Use matrix notation to write down the 2×2 system given by

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

SOLUTION: In this case, the matrix of coefficients and the unknown vector have the form

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

This is an homogeneous system, so the source vector $\boldsymbol{g} = \boldsymbol{0}$. The differential equation can be written as follows,

$$\begin{aligned} x_1' &= x_1 - x_2 \\ x_2' &= -x_1 + x_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x}' = A\boldsymbol{x}.$$

◁

EXAMPLE 5.1.4: Find the explicit expression for the linear system $\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \boldsymbol{g}(t) = \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

SOLUTION: The 2×2 linear system is given by

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{3t} \end{bmatrix}, \quad \Leftrightarrow \quad \begin{aligned} x_1' &= x_1 + 3x_2 + e^t, \\ x_2' &= 3x_1 + x_2 + 2e^{3t}. \end{aligned}$$

◁

EXAMPLE 5.1.5: Show that the vector-valued functions $\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$ are solutions to the 2×2 linear system $\boldsymbol{x}' = A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

SOLUTION: We compute the left-hand side and the right-hand side of the differential equation above for the function $\boldsymbol{x}^{(1)}$ and we see that both side match, that is,

$$A\boldsymbol{x}^{(1)} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} e^{2t} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}; \quad \boldsymbol{x}^{(1)\prime} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \left(e^{2t}\right)' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} 2\, e^{2t},$$

so we conclude that $\boldsymbol{x}^{(1)\,\prime} = A\boldsymbol{x}^{(1)}$. Analogously,

$$A\boldsymbol{x}^{(2)} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} e^{-t} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}; \quad \boldsymbol{x}^{(2)\,\prime} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(e^{-t}\right)' = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t},$$

so we conclude that $\boldsymbol{x}^{(2)\,\prime} = A\boldsymbol{x}^{(2)}$. ◁

EXAMPLE 5.1.6: Find the explicit expression of the most general 3×3 homogeneous linear differential system.

SOLUTION: This is a system of the form $\boldsymbol{x}' = A(t)\,\boldsymbol{x}$, with A beaing a 3×3 matrix. Therefore, we need to find functions x_1, x_2, and x_3 solutions of

$$\begin{aligned} x_1' &= a_{11}(t)\,x_1 + a_{12}(t)\,x_2 + a_{13}(t)\,x_3 \\ x_2' &= a_{21}(t)\,x_1 + a_{22}(t)\,x_2 + a_{13}(t)\,x_3 \\ x_3' &= a_{31}(t)\,x_1 + a_{32}(t)\,x_2 + a_{33}(t)\,x_3. \end{aligned}$$

◁

5.1.2. **Order Transformations.** We present two results that make use of 2×2 linear systems. The first result transforms any second order linear equation into a 2×2 first order linear system. The second result is kind of a converse. It transforms any 2×2 first order, linear, constant coefficients system into a second order linear differential equation. We start with our first result.

Theorem 5.1.2 (First Order Reduction). *A function y solves the second order equation*

$$y'' + p(t)\,y' + q(t)\,y = g(t), \tag{5.1.2}$$

iff the functions $x_1 = y$ and $x_2 = y'$ are solutions to the 2×2 first order differential system

$$x_1' = x_2, \tag{5.1.3}$$

$$x_2' = -q(t)\,x_1 - p(t)\,x_2 + g(t). \tag{5.1.4}$$

Proof of Theorem 5.1.2:
($\Rightarrow$) Given a solution y of Eq. (5.1.2), introduce the functions $x_1 = y$ and $x_2 = y'$. Therefore Eq. (5.1.3) holds, due to the relation

$$x_1' = y' = x_2,$$

Also Eq. (5.1.4) holds, because of the equation

$$x_2' = y'' = -q(t)\,y - p(t)\,y' + g(t) \quad \Rightarrow \quad x_2'' = -q(t)\,x_1 - p(t)\,x_2 + g(t).$$

($\Leftarrow$) Differentiate Eq. (5.1.3) and introduce the result into Eq. (5.1.4), that is,

$$x_1'' = x_2' \quad \Rightarrow \quad x_1'' = -q(t)\,x_1 - p(t)\,x_1' + g(t).$$

Denoting $y = x_1$, we obtain,

$$y'' + p(t)\,y' + q(t)\,y = g(t).$$

This establishes the Theorem. □

EXAMPLE 5.1.7: Express as a first order system the second order equation

$$y'' + 2y' + 2y = \sin(at).$$

SOLUTION: Introduce the new unknowns

$$x_1 = y, \quad x_2 = y' \quad \Rightarrow \quad x_1' = x_2.$$

Then, the differential equation can be written as

$$x_2' + 2x_2 + 2x_1 = \sin(at).$$

We conclude that

$$x_1' = x_2, \qquad x_2' = -2x_1 - 2x_2 + \sin(at).$$

◁

The transformation of a 2×2 first order system into a second order equation given in Theorem 5.1.2 can be generalized to any 2×2 constant coefficient linear differential system.

Theorem 5.1.3 (Second Order Reduction). *Any* $2{\times}2$ *constant coefficients linear system* $\boldsymbol{x}' = A\,\boldsymbol{x}$, *with* $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, *can be written as the second order equation for* x_1 *given by*

$$x_1'' - \operatorname{tr}(A)\,x_1' + \det(A)\,x_1 = 0. \tag{5.1.5}$$

Proof of Theorem 5.1.3: Denoting $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the system has the form

$$x_1' = a_{11}\,x_1 + a_{12}\,x_2 \tag{5.1.6}$$

$$x_2' = a_{21}\,x_1 + a_{22}\,x_2. \tag{5.1.7}$$

Compute the derivative of the first equation,

$$x_1'' = a_{11}\,x_1' + a_{12}\,x_2'.$$

Use Eq. (5.1.7) to replace x_2' on the right-hand side above,

$$x_1'' = a_{11}\,x_1' + a_{12}\big(a_{21}\,x_1 + a_{22}\,x_2\big).$$

Finally, replace the term with x_2 above using Eq. (5.1.8), that is,

$$x_1'' = a_{11}\,x_1' + a_{12}a_{21}\,x_1 + a_{12}a_{22}\,\frac{\big(x_1' - a_{11}\,x_1\big)}{a_{12}}.$$

A simple cancellation and reorganization of terms gives the equation,

$$x_1'' = (a_{11} + a_{22})\,x_1' + (a_{12}a_{21} - a_{11}a_{22})\,x_1.$$

Recalling that $\operatorname{tr}(A) = a_{11} + a_{22}$, and $\det(A) = a_{11}a_{22} - a_{12}a_{21}$, we get

$$x_1'' - \operatorname{tr}(A)\,x_1' + \det(A)\,x_1 = 0.$$

This establishes the Theorem. □

Remark: The component x_2 satisfies exactly the same equation as x_1,

$$x_2'' - \operatorname{tr}(A)\,x_2' + \det(A)\,x_2 = 0. \tag{5.1.8}$$

The proof is the analogous to the one to get the equation for x_1. There is a nice proof to get both equations, for x_1 and x_2, at the same time. It is based in the identity that holds for any 2×2 matrix,

$$A^2 - \operatorname{tr}(A)\,A + \det(A)\,I = 0.$$

This identity is the particular case $n = 2$ of the Cayley-Hamilton Theorem, which holds for $n \times n$ matrices. If we use this identity on the equation for $\boldsymbol{x}''$ we get the equation in Theorem 5.1.3 but for both components x_1 and x_2, because

$$\boldsymbol{x}'' = \big(A\,\boldsymbol{x}\big)' = A\,\boldsymbol{x}' = A^2\,\boldsymbol{x} = \operatorname{tr}(A)\,A\boldsymbol{x} - \det(A) I\boldsymbol{x}.$$

Recalling that $A\,\boldsymbol{x} = \boldsymbol{x}'$, and $I\boldsymbol{x} = \boldsymbol{x}$, we get the vector equation

$$\boldsymbol{x}'' - \operatorname{tr}(A)\,\boldsymbol{x}' + \det(A)\,\boldsymbol{x} = \boldsymbol{0}.$$

The first component of this equation is Eq. (5.1.5), the second component is Eq. (5.1.8)

Example 5.1.8: Express as a single second order equation the 2×2 system and solve it,

$$x_1' = -x_1 + 3x_2,$$
$$x_2' = x_1 - x_2.$$

Solution: Instead of using the result from Theorem 5.1.3, we solve this problem following the proof of that theorem. But instead of working with x_1, we work with x_2. We start computing x_1 from the second equation: $x_1 = x_2' + x_2$. We then introduce this expression into the first equation,

$$(x_2' + x_2)' = -(x_2' + x_2) + 3x_2 \quad \Rightarrow \quad x_2'' + x_2' = -x_2' - x_2 + 3x_2,$$

so we obtain the second order equation

$$x_2'' + 2x_2' - 2x_2 = 0.$$

We solve this equation with the methods studied in Chapter 2, that is, we look for solutions of the form $x_2(t) = e^{rt}$, with r solution of the characteristic equation

$$r^2 + 2r - 2 = 0 \quad \Rightarrow \quad r_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4+8}\right] \quad \Rightarrow \quad r_{\pm} = -1 \pm \sqrt{3}.$$

Therefore, the general solution to the second order equation above is

$$x_2 = c_+ \, e^{(1+\sqrt{3})t} + c_- \, e^{(1-\sqrt{3})t}, \qquad c_+, \; c_- \in \mathbb{R}.$$

Since x_1 satisfies the same equation as x_2, we obtain the same general solution

$$x_1 = \tilde{c}_+ \, e^{(1+\sqrt{3})t} + \tilde{c}_- \, e^{(1-\sqrt{3})t}, \qquad \tilde{c}_+, \; \tilde{c}_- \in \mathbb{R}.$$

◁

5.1.3. The Initial Value Problem.

This notion for linear systems is similar to initial value problems for single differential equations. In the case of an $n \times n$ first order system we need n initial conditions, one for each unknown function, which are collected in an n-vector.

Definition 5.1.4. *An* ***Initial Value Problem*** *for an $n \times n$ linear differential system is the following: Given an $n \times n$ matrix-valued function A, and an n-vector-valued function $\boldsymbol{b}$, a real constant t_0, and an n-vector $\boldsymbol{x}_0$, find an n-vector-valued function $\boldsymbol{x}$ solution of*

$$\boldsymbol{x}' = A(t)\, \boldsymbol{x} + \boldsymbol{b}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0.$$

Remark: The initial condition vector $\boldsymbol{x}_0$ represents n conditions, one for each component of the unknown vector $\boldsymbol{x}$.

Example 5.1.9: Write down explicitly the initial value problem for $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ given by

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Solution: This is a 2×2 system in the unknowns x_1, x_2, with two linear equations

$$x_1' = x_1 + 3x_2$$
$$x_2' = 3x_1 + x_2,$$

and the initial conditions $x_1(0) = 2$ and $x_2(0) = 3$. ◁

The main result about existence and uniqueness of solutions to an initial value problem for a linear system is also analogous to Theorem 2.1.2

Theorem 5.1.5 (Variable Coefficients). *If the functions A and $\boldsymbol{b}$ are continuous on a closed interval $I \subset \mathbb{R}$, $t_0 \in I$, and $\boldsymbol{x}_0$ are any constants, then there exists a unique solution $\boldsymbol{x}$ to the initial value problem*

$$\boldsymbol{x}' = A(t)\,\boldsymbol{x} + \boldsymbol{b}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{5.1.9}$$

Remark: The fixed point argument used in the proof of Picard-Lindelöf's Theorem 1.6.2 can be extended to prove Theorem 5.1.5. This proof will be presented later on.

5.1.4. **Homogeneous Systems.** Solutions to a linear homogeneous differential system satisfy the superposition property: Given two solutions of the homogeneous system, their linear combination is also a solution to that system.

Theorem 5.1.6 (Superposition). *If the n-vector-valued functions $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}$ are solutions of $\boldsymbol{x}^{(1)\prime} = A(t)\,\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)\prime} = A(t)\,\boldsymbol{x}^{(2)}$, then any the linear combination $\boldsymbol{x} = a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}$, for all $a, b \in \mathbb{R}$, is also solution of $\boldsymbol{x}' = A\,\boldsymbol{x}$.*

Remark: This Theorem contains two particular cases:

(a) $a = b = 1$: If $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions of an homogeneous linear system, so is $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$.
(b) $b = 0$ and a arbitrary: If $\boldsymbol{x}^{(1)}$ is a solution of an homogeneous linear system, so is $a\boldsymbol{x}^{(1)}$.

Proof of Theorem 5.1.6: We check that the function $\boldsymbol{x} = a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}$ is a solution of the differential equation in the Theorem. Indeed, since the derivative of a vector-valued function is a linear operation, we get

$$\boldsymbol{x}' = \big(a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}\big)' = a\,\boldsymbol{x}^{(1)\,\prime} + b\,\boldsymbol{x}^{(2)\,\prime}.$$

Replacing the differential equation on the right-hand side above,

$$\boldsymbol{x}' = a\,A\boldsymbol{x}^{(1)} + b\,A\boldsymbol{x}^{(2)}.$$

The matrix-vector product is a linear operation, $A\big(a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}\big) = a\,A\boldsymbol{x}^{(1)} + b\,A\boldsymbol{x}^{(2)}$, hence,

$$\boldsymbol{x}' = A\big(a\boldsymbol{x}^{(1)} + b\boldsymbol{x}^{(2)}\big) \quad \Rightarrow \quad \boldsymbol{x}' = A\boldsymbol{x}.$$

This establishes the Theorem. $\square$

EXAMPLE 5.1.10: Verify that $\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$ and $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$ are solutions to the homogeneous linear system

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}.$$

SOLUTION: The function $\boldsymbol{x}^{(1)}$ is solution to the differential equation, since

$$\boldsymbol{x}^{(1)\,\prime} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}, \qquad A\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} e^{-2t} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}.$$

We then conclude that $\boldsymbol{x}^{(1)\,\prime} = A\boldsymbol{x}^{(1)}$. Analogously, the function $\boldsymbol{x}^{(2)}$ is solution to the differential equation, since

$$\boldsymbol{x}^{(2)\,\prime} = 4\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}, \qquad A\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} e^{4t} = 4\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}.$$

We then conclude that $\boldsymbol{x}^{(2)\,\prime} = A\boldsymbol{x}^{(2)}$. To show that $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}$ is also a solution we could use the linearity of the matrix-vector product, as we did in the proof of the Theorem 5.1.6. Here we choose the straightforward, although more obscure, calculation: On the one hand,

$$\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)} = \begin{bmatrix} e^{-2t} - e^{4t} \\ e^{-2t} + e^{4t} \end{bmatrix} \quad \Rightarrow \quad \big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big)' = \begin{bmatrix} -2e^{-2t} - 4e^{4t} \\ -2e^{-2t} + 4e^{4t} \end{bmatrix}.$$

On the other hand,

$$A\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big) = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} - e^{4t} \\ e^{-2t} + e^{4t} \end{bmatrix} = \begin{bmatrix} e^{-2t} - e^{4t} - 3e^{-2t} - 3e^{4t} \\ -3e^{-2t} + 3e^{4t} + e^{-2t} + e^{4t} \end{bmatrix},$$

that is,

$$A\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big) = \begin{bmatrix} -2e^{-2t} - 4e^{4t} \\ -2e^{-2t} + 4e^{4t} \end{bmatrix}.$$

We conclude that $\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big)' = A\big(\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)}\big)$. $\lhd$

We now introduce the notion of a linearly dependent and independent set of functions.

Definition 5.1.7. *A set of n vector-valued functions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is called* ***linearly dependent*** *on an interval $I \in \mathbb{R}$ iff for all $t \in I$ there exist constants $c_1, \cdots, c_n$, not all of them zero, such that it holds*

$$c_1\, \boldsymbol{x}^{(1)}(t) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t) = \boldsymbol{0}.$$

A set of n vector-valued functions is called ***linearly independent*** *on I iff the set is not linearly dependent.*

Remark: This notion is a generalization of Def. 2.1.6 from two functions to n vector-valued functions. For every value of $t \in \mathbb{R}$ this definition agrees with the definition of a set of linearly dependent vectors given in Linear Algebra and reviewed in the Appendices.

We now generalize Theorem 2.1.7 to linear systems. If you know a linearly independent set of n solutions to an $n \times n$ first order, linear, homogeneous system, then you actually know all possible solutions to that system, since any other solution is just a linear combination of the previous n solutions.

Theorem 5.1.8 (General Solution). *If $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a linearly independent set of solutions of the $n \times n$ system $\boldsymbol{x}' = A\,\boldsymbol{x}$, where A is a continuous matrix-valued function, then there exist unique constants $c_1, \cdots, c_n$ such that every solution $\boldsymbol{x}$ of the differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ can be written as the linear combination*

$$\boldsymbol{x}(t) = c_1\, \boldsymbol{x}^{(1)}(t) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t). \tag{5.1.10}$$

Before we present a sketch of the proof for Theorem 5.1.8, it is convenient to state the following the definitions, which come out naturally from Theorem 5.1.8.

Definition 5.1.9.

(a) The set of functions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a ***fundamental set of solutions*** *of the equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ iff it holds that $\boldsymbol{x}^{(i)\prime} = A\,\boldsymbol{x}^{(i)}$, for $i = 1, \cdots, n$, and the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is linearly independent.*

(b) The ***general solution*** *of the homogeneous equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ denotes any vector-valued function $\boldsymbol{x}_{\text{gen}}$ that can be written as a linear combination*

$$\boldsymbol{x}_{\text{gen}}(t) = c_1\, \boldsymbol{x}^{(1)}(t) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t),$$

where $\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}$ are the functions in any fundamental set of solutions of $\boldsymbol{x}' = A\,\boldsymbol{x}$, while $c_1, \cdots, c_n$ are arbitrary constants.

Remark: The names above are appropriate, since Theorem 5.1.8 says that knowing the n functions of a fundamental set of solutions is equivalent to knowing all solutions to the homogeneous linear differential system.

EXAMPLE 5.1.11: Show that the set of functions $\left\{ \boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t},\ \boldsymbol{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} \right\}$ is a fundamental set of solutions to the system $\boldsymbol{x}' = A\boldsymbol{x}$, where $A = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$.

SOLUTION: In Example 5.1.10 we have shown that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are solutions to the differential equation above. We only need to show that these two functions form a linearly independent set. That is, we need to show that the only constants c_1, c_2 solutions of the equation below, for all $t \in \mathbb{R}$, are $c_1 = c_2 = 0$, where

$$\boldsymbol{0} = c_1\, \boldsymbol{x}^{(1)} + c_2\, \boldsymbol{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = X(t)\, \boldsymbol{c},$$

where $X(t) = \big[\boldsymbol{x}^{(1)}(t),\ \boldsymbol{x}^{(2)}(t)\big]$ and $\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Using this matrix notation, the linear system for c_1, c_2 has the form

$$X(t)\, \boldsymbol{c} = \boldsymbol{0}.$$

We now show that matrix $X(t)$ is invertible for all $t \in \mathbb{R}$. This is the case, since its determinant is

$$\det\big(X(t)\big) = \begin{vmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{vmatrix} = e^{2t} + e^{2t} = 2\, e^{2t} \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

Since $X(t)$ is invertible for $t \in \mathbb{R}$, the only solution for the linear system above is $\boldsymbol{c} = \boldsymbol{0}$, that is, $c_1 = c_2 = 0$. We conclude that the set $\big\{\boldsymbol{x}^{(1)},\ \boldsymbol{x}^{(2)}\big\}$ is linearly independent, so it is a fundamental set of solution to the differential equation above. ◁

Proof of Theorem 5.1.8: The superposition property in Theorem 5.1.6 says that given any set of solutions $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ of the differential equation $\boldsymbol{x}' = A\, \boldsymbol{x}$, the linear combination $\boldsymbol{x}(t) = c_1\, \boldsymbol{x}^{(1)}(t) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t)$ is also a solution. We now must prove that, in the case that $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is linearly independent, every solution of the differential equation is included in this linear combination.

Let now $\boldsymbol{x}$ be any solution of the differential equation $\boldsymbol{x}' = A\, \boldsymbol{x}$. The uniqueness statement in Theorem 5.1.5 implies that this is the only solution that at t_0 takes the value $\boldsymbol{x}(t_0)$. This means that the initial data $\boldsymbol{x}(t_0)$ parametrizes all solutions to the differential equation. We now try to find the constants $\{c_1, \cdots, c_n\}$ solutions of the algebraic linear system

$$\boldsymbol{x}(t_0) = c_1\, \boldsymbol{x}^{(1)}(t_0) + \cdots + c_n\, \boldsymbol{x}^{(n)}(t_0).$$

Introducing the notation

$$X(t) = \big[\boldsymbol{x}^{(1)}(t), \cdots, \boldsymbol{x}^{(n)(t)}\big], \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

the algebraic linear system has the form

$$\boldsymbol{x}(t_0) = X(t_0)\, \boldsymbol{c}.$$

This algebraic system has a unique solution $\boldsymbol{c}$ for every source $\boldsymbol{x}(t_0)$ iff the matrix $X(t_0)$ is invertible. This matrix is invertible iff $\det\big(X(t_0)\big) \neq 0$. The generalization of Abel's Theorem to systems, Theorem 5.1.11, says that $\det\big(X(t_0)\big) \neq 0$ iff the set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ is a fundamental set of solutions to the differential equation. This establishes the Theorem. □

Example 5.1.12: Find the general solution to differential equation in Example 5.1.5 and then use this general solution to find the solution of the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \qquad A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}.$$

Solution: From Example 5.1.5 we know that the general solution of the differential equation above can be written as

$$\boldsymbol{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}.$$

Before imposing the initial condition on this general solution, it is convenient to write this general solution using a matrix-valued function, X, as follows

$$\boldsymbol{x}(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = X(t)\,\boldsymbol{c},$$

where we introduced the solution matrix and the constant vector, respectively,

$$X(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The initial condition fixes the vector $\boldsymbol{c}$, that is, its components c_1, c_2, as follows,

$$\boldsymbol{x}(0) = X(0)\,\boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{c} = \big[X(0)\big]^{-1} \boldsymbol{x}(0).$$

Since the solution matrix X at $t = 0$ has the form,

$$X(0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \big[X(0)\big]^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

introducing $\big[X(0)\big]^{-1}$ in the equation for $\boldsymbol{c}$ above we get

$$\boldsymbol{c} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} c_1 = -1, \\ c_2 = 3. \end{cases}$$

We conclude that the solution to the initial value problem above is given by

$$\boldsymbol{x}(t) = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}.$$

◁

5.1.5. **The Wronskian and Abel's Theorem.** From the proof of Theorem 5.1.8 above we see that it is convenient to introduce the notion of solution matrix and Wronskian of a set of n solutions to an $n \times n$ linear differential system,

Definition 5.1.10.

*(a) The **solution matrix** of any set $\{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\}$ of solutions to a differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ is the $n \times n$ matrix-valued function*

$$X(t) = \big[\boldsymbol{x}^1(t), \cdots, \boldsymbol{x}^n(t)\big]. \tag{5.1.11}$$

*X is called a **fundamental matrix** iff the set $\{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^n\}$ is a fundamental set.*

*(b) The **Wronskian** of the set $\{\boldsymbol{x}^1, \cdots, \boldsymbol{x}^n\}$ is the function $W(t) = \det\big(X(t)\big)$.*

Remark: A fundamental matrix provides a more compact way to write down the general solution of a differential equation. The general solution in Eq. (5.1.10) can be rewritten as

$$\boldsymbol{x}_{\text{gen}}(t) = c_1 \boldsymbol{x}^1(t) + \cdots + c_n \boldsymbol{x}^n(t) = \big[\boldsymbol{x}^1(t), \cdots, \boldsymbol{x}^n(t)\big] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = X(t)\,\boldsymbol{c}, \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

This is a more compact notation for the general solution,

$$\boldsymbol{x}_{\text{gen}}(t) = X(t)\,\boldsymbol{c}. \tag{5.1.12}$$

Remark: Consider the case that a linear system is a first order reduction of a second order linear homogeneous equation, $y'' + a_1 y' + a_0 y = 0$, that is,

$$x_1' = x_2, \qquad x_2' = -a_0 x_1 - a_1 x_2.$$

In this case, the Wronskian defined here coincides with the definition given in Sect. 2.1. The proof is simple. Suppose that y_1, y_2 are fundamental solutions of the second order equation, then the vector-valued functions

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}, \qquad \boldsymbol{x}^{(2)} = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix},$$

are solutions of the first order reduction system. Then holds,

$$W = \det\big([\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}]\big) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W_{y_1 y_2}.$$

EXAMPLE 5.1.13: Find two fundamental matrices for the linear homogeneous system in Example 5.1.10.

SOLUTION: One fundamental matrix is simple to find, is it constructed with the solutions given in Example 5.1.10, that is,

$$X = \big[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\big] \quad \Rightarrow \quad X(t) = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix}.$$

A second fundamental matrix can be obtained multiplying by any non-zero constant each solution above. For example, another fundamental matrix is

$$\tilde{X} = \big[2\boldsymbol{x}^{(1)}, 3\boldsymbol{x}^{(2)}\big] \quad \Rightarrow \quad \tilde{X}(t) = \begin{bmatrix} 2e^{-2t} & -3e^{4t} \\ 2e^{-2t} & 3e^{4t} \end{bmatrix}.$$

◁

EXAMPLE 5.1.14: Compute the Wronskian of the vector-valued functions given in Example 5.1.10, that is, $\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$.

SOLUTION: The Wronskian is the determinant of the solution matrix, with the vectors placed in any order. For example, we can choose the order $\big[\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\big]$. If we choose the order $\big[\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(1)}\big]$, this second Wronskian is the negative of the first one. Choosing the first order we get,

$$W(t) = \det\big([\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}]\big) = \begin{vmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{vmatrix} = e^{-2t}\, e^{4t} + e^{-2t}\, e^{4t}.$$

We conclude that $W(t) = 2e^{2t}$. ◁

EXAMPLE 5.1.15: Show that the set of functions $\left\{ \boldsymbol{x}^{(1)} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix},\ \boldsymbol{x}^{(2)} = \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} \right\}$ is linearly independent for all $t \in \mathbb{R}$.

SOLUTION: We compute the determinant of the matrix $X(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix}$, that is,

$$w(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -2e^{2t} - 2e^{2t} \quad \Rightarrow \quad w(t) = -4e^{2t} \neq 0 \quad t \in \mathbb{R}.$$

◁

We now generalize Abel's Theorem 2.1.14 from a single equation to an $n\times n$ linear system.

Theorem 5.1.11 (Abel). *The Wronskian function $W = \det\big(X(t)\big)$ of a solution matrix $X(t) = \big[\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\big]$ of the linear system $\boldsymbol{x}' = A(t)\boldsymbol{x}$, where A is an $n \times n$ continuous matrix-valued function on a domain $I \subset \mathbb{R}$, is given by*

$$W(t) = W(t_0)\, e^{\alpha(t)}, \qquad \alpha(t) = \int_{t_0}^{t} \operatorname{tr}\big(A(\tau)\big)\, d\tau.$$

where $\operatorname{tr}(A)$ is the trace of A and t_0 is any point in I.

Remarks:

(a) In the case of a constant matrix A, the equation above for the Wronskian reduces to

$$W(t) = W(t_0)\, e^{\operatorname{tr}(A)\,(t-t_0)},$$

(b) The Wronskian function vanishes at a single point iff it vanishes identically for all $t \in I$.

(c) A consequence of (b): n solutions to the system $\boldsymbol{x}' = A(t)\boldsymbol{x}$ are linearly independent at the initial time t_0 iff they are linearly independent for every time $t \in I$.

Proof of Theorem 5.1.11: The proof is based in an identity satisfied by the determinant of certain matrix-valued functions. The proof of this identity is quite involved, so we do not provide it here. The identity is the following: Every $n \times n$, differentiable, invertible, matrix-valued function Z, with values $Z(t)$ for $t \in \mathbb{R}$, satisfies the identity:

$$\frac{d}{dt}\det(Z) = \det(Z)\operatorname{tr}\Big(Z^{-1}\frac{d}{dt}Z\Big).$$

We use this identity with any fundamental matrix $X = \big[\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(n)}\big]$ of the linear homogeneous differential system $\boldsymbol{x}' = A\boldsymbol{x}$. Recalling that the Wronskian $w(t) = \det\big(X(t)\big)$, the identity above says,

$$W'(t) = W(t)\operatorname{tr}\big[X^{-1}(t)\, X'(t)\big].$$

We now compute the derivative of the fundamental matrix,

$$X' = \big[\boldsymbol{x}^{(1)\,\prime}, \cdots, \boldsymbol{x}^{(n)\,\prime}\big] = \big[A\boldsymbol{x}^{(1)}, \cdots, A\boldsymbol{x}^{(n)}\big] = AX,$$

where the equation on the far right comes from the definition of matrix multiplication. Replacing this equation in the Wronskian equation we get

$$W'(t) = W(t)\operatorname{tr}\big(X^{-1}AX\big) = W(t)\operatorname{tr}\big(X\,X^{-1}\,A\big) = W(t)\operatorname{tr}(A),$$

where in the second equation above we used a property of the trace of three matrices: $\operatorname{tr}(ABC) = \operatorname{tr}(CAB) = \operatorname{tr}(BCA)$. Therefore, we have seen that the Wronskian satisfies the equation

$$W'(t) = \operatorname{tr}\big[A(t)\big]\, W(t). \tag{5.1.13}$$

This is a linear differential equation of a single function $W : \mathbb{R} \to \mathbb{R}$. We integrate it using the integrating factor method from Section 1.2. The result is

$$W(t) = W(t_0)\, e^{\alpha(t)}, \qquad \alpha(t) = \int_{t_0}^{t} \operatorname{tr}\big[A(\tau)\big]\, d\tau.$$

This establishes the Theorem. □

Example 5.1.16: Show that the Wronskian of the fundamental matrix constructed with the solutions given in Example 5.1.3 satisfies Eq. (5.1.13) above.

SOLUTION: In Example 5.1.5 we have shown that the vector-valued functions $\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$ and $\boldsymbol{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$ are solutions to the system $\boldsymbol{x}' = A\boldsymbol{x}$, where $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$. The matrix

$$X(t) = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{bmatrix}$$

is a fundamental matrix of the system, since its Wronskian is non-zero,

$$W(t) = \begin{vmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{vmatrix} = 4e^{t} - e^{t} \quad \Rightarrow \quad W(t) = 3e^{t}.$$

We need to compute the right-hand side and the left-hand side of Eq. (5.1.13) and verify that they coincide. We start with the left-hand side,

$$W'(t) = 3e^{t} = W(t).$$

The right-hand side is

$$\operatorname{tr}(A)\, W(t) = (3-2)\, W(t) = W(t).$$

Therefore, we have shown that $W(t) = \operatorname{tr}(A)\, W(t)$. ◁

5.1.6. **Exercises.**

5.1.1.- .

5.1.2.- .

5.2. Constant Coefficients Diagonalizable Systems

Explicit formulas for the solutions of linear differential systems can be found for constant coefficient systems. We find such solution formulas for diagonalizable systems. In this case we transform the originally coupled system into a decoupled system. We then solve the decoupled system and transform back to the original variables. We then arrive at a solution of the original system written in terms of the eigenvalues and eigenvectors of the system coefficient matrix.

5.2.1. **Decoupling the systems.**

The equations in a system of differential equations are usually coupled. In the 2×2 system in Example 5.2.1 below, one must know the function x_2 in order to integrate the first equation to obtain the function x_1. Similarly, one has to know function x_1 to integrate the second equation to get function x_2. The system is coupled; one cannot integrate one equation at a time. One must integrate the whole system together. However, certain coupled differential systems can be decoupled. Such systems are called diagonalizable, and the example below is one of these.

Example 5.2.1: Find functions x_1, x_2 solutions of the first order, 2×2, constant coefficients, homogeneous differential system

$$x_1' = x_1 - x_2,$$
$$x_2' = -x_1 + x_2.$$

Solution: The main idea to solve this system comes from the following observation. If we add up the two equations equations, and if we subtract the second equation from the first, we obtain, respectively,

$$(x_1 + x_2)' = 0, \qquad (x_1 - x_2)' = 2(x_1 - x_2).$$

To understand the consequences of what we have done, let us introduce the new unknowns $v = x_1 + x_2$, and $w = x_1 - x_2$, and re-write the equations above with these new unknowns,

$$v' = 0, \qquad w' = 2w.$$

We have decoupled the original system. The equations for x_1 and x_2 are coupled, but we have found a linear combination of the equations such that the equations for v and w are not coupled. We now solve each equation independently of the other.

$$v' = 0 \quad \Rightarrow \quad v = c_1,$$
$$w' = 2w \quad \Rightarrow \quad w = c_2 e^{2t},$$

with c_1, $c_2 \in \mathbb{R}$. Having obtained the solutions for the decoupled system, we now transform back the solutions to the original unknown functions. From the definitions of v and w we see that

$$x_1 = \frac{1}{2}\,(v + w), \qquad x_2 = \frac{1}{2}\,(v - w).$$

We conclude that for all c_1, $c_2 \in \mathbb{R}$ the functions x_1, x_2 below are solutions of the 2×2 differential system in the example, namely,

$$x_1(t) = \frac{1}{2}\,(c_1 + c_2 e^{2t}), \qquad x_2(t) = \frac{1}{2}\,(c_1 - c_2 e^{2t}). \qquad \triangleleft$$

Let us review what we did in the example above. The equations for x_1 and x_2 are coupled, so we found an appropriate linear combination of the equations and the unknowns such that the equations for the new unknown functions, u and v, are decoupled. We integrated each equation independently of the other, and we finally transformed back to the original unknowns x_1 and x_2. The key step is to find the transformation from the x_1, x_2 into the u, v.

For general systems this transformation may not exists. It exists, however, for a particular type of systems, called diagonalizable. We start reviewing the concept of a diagonalizable matrix.

Definition 5.2.1. *An $n \times n$ matrix A is called* ***diagonalizable*** *iff there exists an invertible matrix P and a diagonal matrix D such that*

$$A = PDP^{-1}.$$

Remark: This and other concepts of linear algebra can be reviewed in Chapter 8.

Example 5.2.2: Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable, where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Solution: That matrix P is invertible can be verified by computing its determinant, $\det(P) = 1 - (-1) = 2$. Since the determinant is non-zero, P is invertible. Using linear algebra methods one can find out that the inverse matrix is $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Now we only need to verify that PDP^{-1} is indeed A. A straightforward calculation shows

$$\begin{aligned} PDP^{-1} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad PDP^{-1} = A. \end{aligned}$$

$\lhd$

Remark: Not every matrix is diagonalizable. For example the matrix $B = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable. The following result tells us how to find out whether a matrix is diagonalizable or not, and in the former case, how to compute matrices P and D. See Chapter 8 for a proof.

Theorem 5.2.2 (Diagonalizable matrices). *An $n \times n$ matrix A is diagonalizable iff this matrix A has a linearly independent set of n eigenvectors. Furthermore, the decomposition $A = PDP^{-1}$ holds for matrices P and D given by*

$$P = [\boldsymbol{v}^1, \cdots, \boldsymbol{v}^n], \quad D = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big],$$

where $\lambda_i, \boldsymbol{v}^i$, for $i = 1, \cdots, n$, are eigenvalue-eigenvector pairs of matrix A.

Example 5.2.3: Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: We need to find the eigenvalues of matrix A. They are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$. So we first compute

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}.$$

Then we compute its determinant,

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9$$

The roots of the characteristic polynomial are the eigenvalues of matrix A,

$$p(\lambda) = (\lambda - 1)^2 - 9 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_+ = 4, \\ \lambda_- = -2. \end{cases}$$

We have obtained two eigenvalues, so now we compute their associated eigenvectors. The eigenvector $\boldsymbol{v}^+$ associated with the eigenvalue λ_+ is a solution of the equation

$$A\boldsymbol{v}^+ = \lambda_+ \boldsymbol{v}^+ \quad \Leftrightarrow \quad (A - \lambda_+ I)\boldsymbol{v}^+ = \boldsymbol{0}.$$

So we first compute matrix $A - \lambda^+ I$ for $\lambda_+ = 4$, that is,

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

Then we solve for $\boldsymbol{v}^+$ the equation

$$(A - 4I)\boldsymbol{v}^+ = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^+ = v_2^+, \\ v_2^+ \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\boldsymbol{v}^+ = \begin{bmatrix} v_2^+ \\ v_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^+ \quad \Rightarrow \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^+ = 1$. A similar calculation provides the eigenvector $\boldsymbol{v}^-$ associated with the eigenvalue $\lambda_- = -2$, that is, first compute the matrix

$$A + 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

then we solve for $\boldsymbol{v}^-$ the equation

$$(A + 2I)\boldsymbol{v}^- = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^- = -v_2^-, \\ v_2^- \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\boldsymbol{v}^- = \begin{bmatrix} -v_2^- \\ v_2^- \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^- \quad \Rightarrow \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^- = 1$. We therefore conclude that the eigenvalues and eigenvectors of the matrix A above are given by

$$\lambda_+ = 4, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

With these eigenvalues and eigenvectors we construct matrices P and D as in Theorem 5.2.2,

$$P = \begin{bmatrix} \boldsymbol{v}^+, \boldsymbol{v}^- \end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix};$$

$$D = \operatorname{diag}\big[\lambda_+, \lambda_-\big] \quad \Rightarrow \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

We have already shown in Example 5.1.7 that P is invertible and that $A = PDP^{-1}$. ◁

5.2.2. **Eigenvector solution formulas.** Explicit formulas for the solution of linear differential system can be obtained in the case that the system coefficient matrix is diagonalizable.

Definition 5.2.3. *A **diagonalizable differential system** is a differential equation of the form $\boldsymbol{x}' = A\,\boldsymbol{x} + \boldsymbol{g}$, where the coefficient matrix A is diagonalizable.*

For diagonalizable differential systems there is an explicit formula for the solution of the differential equation. The formula includes the eigenvalues and eigenvectors of the coefficient matrix.

Theorem 5.2.4 (Eigenpairs expression). *If the $n \times n$ constant matrix A is diagonalizable, with a set of linearly independent eigenvectors $\big\{\boldsymbol{v}^1, \cdots, \boldsymbol{v}^n\big\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then, the system $\boldsymbol{x}' = A\,\boldsymbol{x}$ has a set of fundamental solutions given by*

$$\big\{\boldsymbol{x}^1(t) = \boldsymbol{v}^1 e^{\lambda_1 t}, \cdots, \boldsymbol{x}^n(t) = \boldsymbol{v}^n e^{\lambda_n t}\big\}. \tag{5.2.1}$$

Furthermore, every initial value problem $\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$, with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, has a unique for every initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$,

$$\boldsymbol{x}(t) = c_1 \boldsymbol{v}^1 e^{\lambda_1 t} + \cdots + c_n \boldsymbol{v}^n e^{\lambda_n t}, \tag{5.2.2}$$

where the constants $c_1, \cdots, c_n$ are solution of the algebraic linear system

$$\boldsymbol{x}_0 = c_1 \boldsymbol{v}^1 e^{\lambda_1 t_0} + \cdots + c_n \boldsymbol{v}^n e^{\lambda_1 t_0}. \tag{5.2.3}$$

Remarks:

(a) We show two proofs of this Theorem. The first one is short but uses Theorem 5.1.8. The second proof is constructive, longer than the first proof, and it makes no use of Theorem 5.1.8.

(b) The second proof follows the same idea presented to solve Example 5.2.1. We decouple the system, we solve the uncoupled system, and we transform back to the original unknowns. The differential system is decoupled when written in the basis of eigenvectors of the coefficient matrix.

First proof of Theorem 5.2.4: Each function $\boldsymbol{x}^i = \boldsymbol{v}^i e^{\lambda_i t}$, for $i = 1, \cdots, n$, is solution of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$, because

$$\boldsymbol{x}^{i\prime} = \lambda_i \boldsymbol{v}^i e^{\lambda_1 t}, \qquad A\boldsymbol{x}^i = A\boldsymbol{v}^i\, e^{\lambda_i t} = \lambda_i \boldsymbol{v}^i e^{\lambda_i t},$$

hence $\boldsymbol{x}^{i\prime} = A\,\boldsymbol{x}^i$. Since A is diagonalizable, the set $\big\{\boldsymbol{x}^1(t) = \boldsymbol{v}^1 e^{\lambda_1 t}, \cdots, \boldsymbol{x}^n(t) = \boldsymbol{v}^n e^{\lambda_n t}\big\}$ is a fundamental set of the system. Therefore, Theorem 5.1.8 says that teh general solution to the system is

$$\boldsymbol{x}(t) = c_1 \boldsymbol{v}^1 e^{\lambda_1 t} + \cdots + c_n \boldsymbol{v}^n e^{\lambda_n t}.$$

The constants $c_1, \cdots, c_n$ are computed by evaluating the equation above at t_0 and recalling the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$. The result is Eq. (5.2.3). This establishes the Theorem. □

Remark: The proof above does not say how one can find out that a function of the form $\boldsymbol{x}^i = \boldsymbol{v}^i e^{\lambda_i t}$ is a solution in the first place. The second proof below constructs the solutions and shows that the solutions are indeed the ones in the first proof.

Second proof of Theorem 5.2.4: Since the coefficient matrix A is diagonalizable, there exist an $n \times n$ invertible matrix P and an $n \times n$ diagonal matrix D such that $A = PDP^{-1}$. Introduce that into the differential equation and multiplying the whole equation by P^{-1},

$$P^{-1}\boldsymbol{x}'(t) = P^{-1}\big(PDP^{-1}\big)\,\boldsymbol{x}(t).$$

Since matrix A is constant, so is P and D. In particular $P^{-1}\boldsymbol{x}' = \big(P^{-1}\boldsymbol{x}\big)'$, hence

$$(P^{-1}\boldsymbol{x})' = D\,\big(P^{-1}\,\boldsymbol{x}\big).$$

Define the unknown function $\boldsymbol{y} = \big(P^{-1}\boldsymbol{x}\big)$. The differential equation is now given by

$$\boldsymbol{y}'(t) = D\,\boldsymbol{y}(t).$$

Since matrix D is diagonal, the system above is a decoupled for the unknown $\boldsymbol{y}$. Transform the initial condition too, that is, $P^{-1}\boldsymbol{x}(t_0) = P^{-1}\boldsymbol{x}_0$. Introduce the notation $\boldsymbol{y}_0 = P^{-1}\boldsymbol{x}_0$, so the initial condition is

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0.$$

Solve the decoupled initial value problem $\boldsymbol{y}'(t) = D\,\boldsymbol{y}(t)$,

$$\left.\begin{aligned} y_1'(t) &= \lambda_1\,y_1(t),\\ &\;\vdots\\ y_n'(t) &= \lambda_n\,y_n(t),\end{aligned}\right\} \quad\Rightarrow\quad \left\{\begin{aligned} y_1(t) &= c_1\,e^{\lambda_1 t},\\ &\;\vdots\\ y_n(t) &= c_n\,e^{\lambda_n t},\end{aligned}\right\} \quad\Rightarrow\quad \boldsymbol{y}(t) = \begin{bmatrix} c_1\,e^{\lambda_1 t}\\ \vdots\\ c_n\,e^{\lambda_n t}\end{bmatrix}.$$

Once $\boldsymbol{y}$ is found, we transform back to $\boldsymbol{x}$,

$$\boldsymbol{x}(t) = P\,\boldsymbol{y}(t) = \big[\boldsymbol{v}^1, \cdots, \boldsymbol{v}^n\big]\begin{bmatrix} c_1\,e^{\lambda_1 t}\\ \vdots\\ c_n\,e^{\lambda_n t}\end{bmatrix} = c_1\,\boldsymbol{v}^1 e^{\lambda_1 t} + \cdots + c_n\,\boldsymbol{v}^n e^{\lambda_n t}.$$

This is Eq. (5.2.2). Evaluating it at t_0 we obtain Eq. (5.2.3). This establishes the Theorem. □

Example 5.2.4: Find the vector-valued function $\boldsymbol{x}$ solution to the differential system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}, \quad A = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}.$$

Solution: First we need to find out whether the coefficient matrix A is diagonalizable or not. Theorem 5.2.2 says that a 2×2 matrix is diagonalizable iff there exists a linearly independent set of two eigenvectors. So we start computing the matrix eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix}(1-\lambda) & 2\\ 2 & (1-\lambda)\end{vmatrix} = (1-\lambda)^2 - 4.$$

The roots of the characteristic polynomial are

$$(\lambda - 1)^2 = 4 \quad\Leftrightarrow\quad \lambda_\pm = 1 \pm 2 \quad\Leftrightarrow\quad \lambda_+ = 3, \quad \lambda_- = -1.$$

The eigenvectors corresponding to the eigenvalue $\lambda_+ = 3$ are the solutions $\boldsymbol{v}^+$ of the linear system $(A - 3I_2)\boldsymbol{v}^+ = \boldsymbol{0}$. To find them, we perform Gauss operations on the matrix

$$A - 3I_2 = \begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix} \to \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} \quad\Rightarrow\quad v_1^+ = v_2^+ \quad\Rightarrow\quad \boldsymbol{v}^+ = \begin{bmatrix}1\\1\end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue $\lambda_- = -1$ are the solutions $\boldsymbol{v}^-$ of the linear system $(A + I_2)\boldsymbol{v}^- = \boldsymbol{0}$. To find them, we perform Gauss operations on the matrix

$$A + I_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1^- = -v_2^- \quad \Rightarrow \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Summarizing, the eigenvalues and eigenvectors of matrix A are following,

$$\lambda_+ = 3, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -1, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Once we have the eigenvalues and eigenvectors of the coefficient matrix, Eq. (5.2.2) gives us the general solution

$$\boldsymbol{x}(t) = c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t},$$

where the coefficients c_+ and c_- are solutions of the initial condition equation

$$c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We conclude that $c_+ = 5/2$ and $c_- = -1/2$, hence

$$\boldsymbol{x}(t) = \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = \frac{1}{2} \begin{bmatrix} 5e^{3t} + e^{-t} \\ 5e^{3t} - e^{-t} \end{bmatrix}.$$

◁

Example 5.2.5: Find the general solution to the 2×2 differential system

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Solution: We start finding the eigenvalues and eigenvectors of the coefficient matrix A. This part of the work was already done in Example 5.2.3. We have found that A has two linearly independent eigenvectors, more precisely,

$$\lambda_+ = 4, \qquad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad \boldsymbol{x}^+(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t},$$

$$\lambda_- = -2, \qquad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad \boldsymbol{x}^-(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

Therefore, the general solution of the differential equation is

$$\boldsymbol{x}(t) = c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \qquad c_+, c_- \in \mathbb{R}.$$

◁

5.2.3. **Alternative solution formulas.** There are several ways to write down the solution found in Theorem 5.2.4. The formula in Eq. (5.2.2) is useful to write down the general solution to the equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ when A diagonalizable. It is a formula easy to remember, you just add all terms of the form $\boldsymbol{v}^i e^{\lambda_i t}$, where λ_i, $\boldsymbol{v}^i$ is any eigenpair of A. But this formula is not the best one to write down solutions to initial value problems. As you can see in Theorem 5.2.4, I did not provide a formula for that. I only said that the constants $c_1, \cdots, c_n$ are the solutions of the algebraic linear system in (5.2.3). But I did not write down the solution for the c's. It is too complicated in this notation, though it is not difficult to do on a particular example, as near the end of Example 5.2.2.

A fundamental matrix, introduced in Eq. (5.1.11), provides a more compact form for the solution of an initial value problem. We have seen this compact notation in Eq. (5.1.12),

$$\boldsymbol{x}(t) = X(t)\,\boldsymbol{c},$$

where we used the fundamental matrix constructed with the fundamental solutions in (5.2.1), and we collected all the c's in a vector,

$$X(t) = \big[\boldsymbol{v}^1 e^{\lambda_1 t}, \cdots, \boldsymbol{v}^n e^{\lambda_n t}\big], \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

The equation from the initial condition is now

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0) = X(t_0)\,\boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{c} = X(t_0)^{-1}\boldsymbol{x}_0,$$

which makes sense, since $X(t)$ is an invertible matrix for all t where it is defined. Using this formula for the constant vector $\boldsymbol{c}$ we get,

$$\boldsymbol{x}(t) = X(t)X(t_0)^{-1}\,\boldsymbol{x}_0.$$

We summarize this result in a statement.

Theorem 5.2.5 (Fundamental matrix expression). *If the $n \times n$ constant matrix A is diagonalizable, with a set of linearly independent eigenvectors $\big\{\boldsymbol{v}^1, \cdots, \boldsymbol{v}^n\big\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then, the initial value problem $\boldsymbol{x}' = A\,\boldsymbol{x}$ with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ has a unique solution given by*

$$\boldsymbol{x}(t) = X(t)X(t_0)^{-1}\,\boldsymbol{x}_0 \tag{5.2.4}$$

where $X(t) = \big[\boldsymbol{v}^1 e^{\lambda_1 t}, \cdots, \boldsymbol{v}^n e^{\lambda_n t}\big]$ is a fundamental matrix of the system.

Remark: Eq. (5.2.4) also holds in the case that the coefficient matrix A is not diagonalizable. In such case the fundamental matrix X is not given by the expression provided in the Theorem. But with an appropriate fundamental matrix, Eq. (5.2.4) still holds.

Example 5.2.6: Find a fundamental matrix for the system below and use it to write down the general solution to the system, where

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution: One way to find a fundamental matrix of a system is to start computing the eigenvalues and eigenvectors of the coefficient matrix. The differential equation in this Example is the same as the one given in Example 5.2.2. In that Example we found out that the eigenvalues and eigenvectors of the coefficient matrix were,

$$\lambda_+ = 3, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -1, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We see that the coefficient matrix is diagonalizable, so with the information above we can construct a fundamental set of solutions,

$$\Big\{\boldsymbol{x}^+(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t},\ \boldsymbol{x}^-(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}\Big\}.$$

From here we construct a fundamental matrix

$$X(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}.$$

Then we have the general solution

$$\boldsymbol{x}_{\text{gen}}(t) = X(t)\boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{x}_{\text{gen}}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} c_+ \\ c_- \end{bmatrix}.$$

◁

Example 5.2.7: Use the fundamental matrix found in Example 5.2.6 to write down the solution to the initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution: In Example 5.2.6 we found the general solution to the differential equation,

$$\boldsymbol{x}_{\text{gen}}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

The initial condition has the form

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \boldsymbol{x}(0) = X(0)\,\boldsymbol{c} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

We need to compute the inverse of matrix $X(0)$,

$$X(0)^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

so we compute the constant vector $\boldsymbol{c}$,

$$\begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

So the solution to the initial value problem is,

$$\boldsymbol{x}(t) = X(t)X(0)^{-1}\boldsymbol{x}(0) \quad \Leftrightarrow \quad \boldsymbol{x}(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

If we compute the matrix on the last equation, explicitly, we get,

$$\boldsymbol{x}(t) = \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

◁

Example 5.2.8: Use a fundamental matrix to write the solution to the initial value problem

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Solution: We known from Example ?? that the general solution to the differential equation above is given by

$$\boldsymbol{x}(t) = c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}, \qquad c_+, c_- \in \mathbb{R}.$$

Equivalently, introducing the fundamental matrix X and the vector $\boldsymbol{c}$ as in Example ??,

$$X(t) = \begin{bmatrix} e^{4t} & -e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

so the general solution can be written as

$$\boldsymbol{x}(t) = X(t)\boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{x}(t) = \begin{bmatrix} e^{4t} & -e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The initial condition is an equation for the constant vector $\boldsymbol{c}$,

$$X(0)\,\boldsymbol{c} = \boldsymbol{x}(0) \quad \Leftrightarrow \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

The solution of the linear system above can be expressed in terms of the inverse matrix

$$\big[X(0)\big]^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

as follows,

$$\boldsymbol{c} = \big[X(0)\big]^{-1} \boldsymbol{x}(0) \quad \Leftrightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

So, the solution to the initial value problem in vector form is given by

$$\boldsymbol{x}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

and using the fundamental matrix above we get

$$\boldsymbol{x}(t) = \begin{bmatrix} e^{4t} & -e^{-2t} \\ e^{4t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{x}(t) = \begin{bmatrix} 3e^{4t} - e^{-2t} \\ 3e^{4t} + e^{-2t} \end{bmatrix}.$$

◁

We saw that the solution to the initial value problem $\boldsymbol{x}' = A\,\boldsymbol{x}$ with $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ can be written using a fundamental matrix X,

$$\boldsymbol{x}(t) = X(t)X(t_0)^{-1}\,\boldsymbol{x}_0.$$

There is an alternative expression to write the solution of the initial value problem above. It makes use of the exponential of a the coefficient matrix.

Theorem 5.2.6 (Exponential expression). *The initial value problem for an $n \times n$ homogeneous, constant coefficients, linear differential system*

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

has a unique solution $\boldsymbol{x}$ for every $t_0 \in \mathbb{R}$ and every n-vector $\boldsymbol{x}_0$, given by

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}_0. \tag{5.2.5}$$

Remarks:

(a) The Theorem holds as it is written, for any constant $n \times n$ matrix A, whether it is diagonalizable or not. But in this Section we provide a proof of the Theorem only in the case that A is diagonalizable.

(b) Eq. 5.2.5 is a nice generalization of the solutions for a single linear homogeneous equations we found in Section 1.1.

Proof of Theorem 5.2.6 for a diagonalizable matrix A: We start with the formula for the fundamental matrix given in Theorem 5.2.5,

$$X(t) = \big[\boldsymbol{v}^1 e^{\lambda_1 t}, \cdots, \boldsymbol{v}^n e^{\lambda_n t}\big] = \big[\boldsymbol{v}^1, \cdots, \boldsymbol{v}^n\big] \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix},$$

The diagonal matrix on the last equation above is the exponential of the diagonal matrix

$$Dt = \operatorname{diag}\big[\lambda_1 t, \cdots, \lambda_n t\big].$$

This is the case, since by the definitions in Chapter 8 we have,

$$e^{Dt} = \sum_{n=0}^{\infty} \frac{(Dt)^n}{n!} = \operatorname{diag}\Big[\sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{n!}, \cdots, \sum_{n=0}^{\infty} \frac{(\lambda_n t)^n}{n!}\Big]$$

which gives us the expression for the exponential of a diagonal matrix,

$$e^{Dt} = \operatorname{diag}\big[e^{\lambda_1 t}, \cdots, e^{\lambda_n t}\big].$$

One more thing, let us denote $P = \big[\boldsymbol{v}^1, \cdots, \boldsymbol{v}^n\big]$, as we did in Chapter 8. If we use these two expressions into the formula for X above, we get

$$X(t) = P\, e^{Dt}.$$

Using properties of invertible matrices, given in Chapter 8 we get the following,

$$X(t_0)^{-1} = \big(Pe^{Dt_0}\big)^{-1} = e^{-Dt_0} P^{-1},$$

where we used that $\big(e^{Dt_0}\big)^{-1} = e^{-Dt_0}$. These manipulations lead us to the formula

$$X(t)X(t_0)^{-1} = Pe^{Dt} e^{-Dt_0} P^{-1} \quad \Rightarrow \quad X(t)X(t_0)^{-1} = Pe^{D(t-t_0)} P^{-1}.$$

The last step of the argument is to relate the equation above with $e^{A(t-t_0)}$. Since A is diagonalizable, $A = PDP^{-1}$, for the matrices P and D defined above. Then,

$$e^{A(t-t_0)} = \sum_{n=0}^{\infty} \frac{A^n (t-t_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{(PDP^{-1})^n (t-t_0)^n}{n!} = P\Big(\sum_{n=0}^{\infty} \frac{D^n (t-t_0)^n}{n!}\Big)P^{-1},$$

and by the calculation we did in the first part of this proof we get,

$$e^{A(t-t_0)} = Pe^{D(t-t_0)} P^{-1}$$

We conclude that $X(t)X(t_0)^{-1} = e^{A(t-t_0)}$, which gives us to the formula

$$\boldsymbol{x}(t) = e^{A(t-t_0)} \boldsymbol{x}_0.$$

This establishes the Theorem. □

Let us take another look at the problem in the Example 5.2.4.

Example 5.2.9: Compute the exponential function e^{At} and use it to express the vector-valued function $\boldsymbol{x}$ solution to the initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Solution: In Example 5.2.6 we found that a fundamental matrix for this system was

$$X(t) = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}.$$

From the proof of the Theorem above we see that

$$X(t)X(t_0)^{-1} = Pe^{D(t-t_0)} P^{-1} = e^{A(t-t_0)}.$$

In this Example $t_0 = 0$ and

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix},$$

so we get $X(t)X(0)^{-1} = Pe^{Dt}P^{-1} = e^{At}$, that is,

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

so we conclude that

$$e^{At} = \frac{1}{2}\begin{bmatrix} (e^{3t}+e^{-t}) & (e^{3t}-e^{-t}) \\ (e^{3t}-e^{-t}) & (e^{3t}+e^{-t}) \end{bmatrix}.$$

The solution to the initial value problem above is,

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 = \frac{1}{2}\begin{bmatrix} (e^{3t}+e^{-t}) & (e^{3t}-e^{-t}) \\ (e^{3t}-e^{-t}) & (e^{3t}+e^{-t}) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad \lhd$$

5.2.4. **Non-homogeneous systems.** We continue our study of diagonalizable linear differential systems with the case of non-homogeneous, continuous sources. The solution of an initial value problem in this case involves again the exponential of the coefficient matrix. The solution for these systems is a generalization of the solution formula for single scalar equations, which is given by Eq. (1.2.7), in the case that the coefficient function a is constant.

Theorem 5.2.7 (Exponential expression). *If the $n \times n$ constant matrix A is diagonalizable and the n-vector-valued function $\boldsymbol{g}$ is constant, then the initial value problem*

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t) + \boldsymbol{g}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0.$$

has a unique solution for every initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ given by

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\Big[\boldsymbol{x}_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)}\,\boldsymbol{g}(\tau)\,d\tau\Big]. \tag{5.2.6}$$

Remark: In the case of an homogeneous system, $\boldsymbol{g} = \boldsymbol{0}$, and we reobtain Eq. (5.2.5). In the case that the coefficient matrix A is invertible and the source function $\boldsymbol{g}$ is constant, the integral in Eq. (5.2.6) can be computed explicitly and the result is

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\big(\boldsymbol{x}_0 - A^{-1}\boldsymbol{g}\big) - A^{-1}\boldsymbol{g}.$$

The expression above is the generalizations for systems of Eq. (1.1.10) for scalar equations.

Proof of Theorem 5.2.7: Since the coefficient matrix A is diagonalizable, there exist an $n\times n$ invertible matrix P and an $n\times n$ diagonal matrix D such that $A = PDP^{-1}$. Introducing this information into the differential equation in Eq. (1.1.10), and then multiplying the whole equation by P^{-1}, we obtain,

$$P^{-1}\boldsymbol{x}'(t) = P^{-1}\big(PDP^{-1}\big)\,\boldsymbol{x}(t) + P^{-1}\boldsymbol{g}(t).$$

Since matrix A is constant, so is P and D. In particular $P^{-1}\boldsymbol{x}' = \big(P^{-1}\boldsymbol{x}\big)'$. Therefore,

$$(P^{-1}\boldsymbol{x})' = D\,\big(P^{-1}\,\boldsymbol{x}\big) + \big(P^{-1}\boldsymbol{g}\big).$$

Introduce the new unknown function $\boldsymbol{y} = \big(P^{-1}\boldsymbol{x}\big)$ and the new source function $\boldsymbol{h} = \big(P^{-1}\boldsymbol{g}\big)$, then the equation above has the form

$$\boldsymbol{y}'(t) = D\,\boldsymbol{y}(t) + \boldsymbol{h}(t).$$

Now transform the initial condition too, that is, $P^{-1}\boldsymbol{x}(t_0) = P^{-1}\boldsymbol{x}_0$. Introducing the notation $\boldsymbol{y}_0 = P^{-1}\boldsymbol{x}_0$, we get the initial condition

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0.$$

Since the coefficient matrix D is diagonal, the initial value problem above for the unknown $\boldsymbol{y}$ is decoupled. That is, expressing $D = \mathrm{diag}[\lambda_1, \cdots, \lambda_n]$, $\boldsymbol{h} = [h_i]$, $\boldsymbol{y}_0 = [y_{0i}]$, and $\boldsymbol{y} = [y_i]$, then for $i = 1, \cdots, n$ holds that the initial value problem above has the form

$$y_i'(t) = \lambda_i\,y_i(t) + h_i(t), \qquad y_i(t_0) = y_{0i}.$$

The solution of each initial value problem above can be found using Eq. (1.2.7), that is,

$$y_i(t) = e^{\lambda_i(t-t_0)}\Big[y_{0i} + \int_{t_0}^t e^{-\lambda_i(\tau-t_0)} h_i(\tau)\,d\tau\Big].$$

We rewrite the equations above in vector notation as follows,

$$\boldsymbol{y}(t) = e^{D(t-t_0)}\Big[\boldsymbol{y}_0 + \int_{t_0}^t e^{-D(\tau-t_0)} \boldsymbol{h}(\tau)\,d\tau\Big],$$

where we recall that $e^{D(t-t_0)} = \mathrm{diag}\big[e^{\lambda_1(t-t_0)}, \cdots, e^{\lambda_n(t-t_0)}\big]$. We now multiply the whole equation by the constant matrix P and we recall that $P^{-1}P = I_n$, then

$$P\boldsymbol{y}(t) = Pe^{D(t-t_0)}\Big[(P^{-1}P)\boldsymbol{y}_0 + (P^{-1}P)\int_{t_0}^t e^{-D(\tau-t_0)}(P^{-1}P)\boldsymbol{h}(\tau)\,d\tau\Big].$$

Recalling that $P\boldsymbol{y} = \boldsymbol{x}$, $P\boldsymbol{y}_0 = \boldsymbol{x}_0$, $P\boldsymbol{h} = \boldsymbol{g}$, and $e^{At} = Pe^{Dt}P^{-1}$, we obtain

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\Big[\boldsymbol{x}_0 + \int_{t_0}^t e^{-A(\tau-t_0)} \boldsymbol{g}(\tau)\,d\tau\Big].$$

This establishes the Theorem. □

We said it in the homogeneous equation and we now say it again. Although the expression for the solution given in Eq. (**??**) looks simple, the exponentials of the coefficient matrix are actually not that simple to compute. We show this in the following example.

Example 5.2.10: Find the vector-valued solution $\boldsymbol{x}$ to the differential system

$$\boldsymbol{x}' = A\,\boldsymbol{x} + \boldsymbol{g}, \quad \boldsymbol{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}, \quad A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix}1\\2\end{bmatrix}.$$

Solution: In Example 5.2.4 we have found the eigenvalues and eigenvectors of the coefficient matrix, and the result is

$$\lambda_1 = 3, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \text{and} \quad \lambda_2 = -1, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix}-1\\1\end{bmatrix}.$$

With this information and Theorem 5.2.2 we obtain that

$$A = PDP^{-1}, \qquad P = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}, \qquad D = \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix},$$

and also that

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}e^{3t} & 0\\0 & e^{-t}\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix},$$

so we conclude that

$$e^{At} = \frac{1}{2}\begin{bmatrix}(e^{3t}+e^{-t}) & (e^{3t}-e^{-t})\\(e^{3t}-e^{-t}) & (e^{3t}+e^{-t})\end{bmatrix} \quad \Rightarrow \quad e^{-At} = \frac{1}{2}\begin{bmatrix}(e^{-3t}+e^{t}) & (e^{-3t}-e^{t})\\(e^{-3t}-e^{t}) & (e^{-3t}+e^{t})\end{bmatrix}.$$

The solution to the initial value problem above is,

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + e^{At}\int_0^t e^{-A\tau}\boldsymbol{g}\,d\tau.$$

Since

$$e^{At}\boldsymbol{x}_0 = \frac{1}{2}\begin{bmatrix}(e^{3t}+e^{-t}) & (e^{3t}-e^{-t})\\(e^{3t}-e^{-t}) & (e^{3t}+e^{-t})\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}5e^{3t}+e^{-t}\\5e^{3t}-e^{-t}\end{bmatrix},$$

in a similar way

$$e^{-A\tau}\boldsymbol{g} = \frac{1}{2}\begin{bmatrix}(e^{-3\tau}+e^{\tau}) & (e^{-3\tau}-e^{\tau})\\(e^{-3\tau}-e^{\tau}) & (e^{-3\tau}+e^{\tau})\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3e^{-3\tau}-e^{\tau}\\3e^{-3\tau}+e^{\tau}\end{bmatrix}.$$

Integrating the last expresion above, we get

$$\int_0^t e^{-A\tau} \boldsymbol{g}\, d\tau = \frac{1}{2} \begin{bmatrix} -e^{-3t} - e^{t} \\ -e^{-3t} + e^{t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, we get

$$\boldsymbol{x}(t) = \frac{1}{2} \begin{bmatrix} 5e^{3t} + e^{-t} \\ 5e^{3t} - e^{-t} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) & (e^{3t} - e^{-t}) \\ (e^{3t} - e^{-t}) & (e^{3t} + e^{-t}) \end{bmatrix} \left[\frac{1}{2} \begin{bmatrix} -e^{-3t} - e^{t} \\ -e^{-3t} + e^{t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right].$$

Multiplying the matrix-vector product on the second term of the left-hand side above,

$$\boldsymbol{x}(t) = \frac{1}{2} \begin{bmatrix} 5e^{3t} + e^{-t} \\ 5e^{3t} - e^{-t} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (e^{3t} + e^{-t}) \\ (e^{3t} - e^{-t}) \end{bmatrix}.$$

We conclude that the solution to the initial value problem above is

$$\boldsymbol{x}(t) = \begin{bmatrix} 3e^{3t} + e^{-t} - 1 \\ 3e^{3t} - e^{-t} \end{bmatrix}.$$

$\triangleleft$

5.2.5. **Exercises.**

5.2.1.- .

5.2.2.- .

5.3. Two-by-Two Constant Coefficients Systems

2×2 linear systems are important not only by themselves but as approximations of more complicated nonlinear systems. They are important by themselves because 2×2 systems are simple enough so their solutions can be computed and classified. But they are non-trivial enough so their solutions describe several situations including exponential decays and oscillations. In this Section we study 2×2 systems in detail and we classify them according the eigenvalues of the coefficient matrix. In a later Chapter we will use them as approximations of more complicated systems.

5.3.1. The diagonalizable case.

Consider a 2×2 constant coefficient, homogeneous linear differential system,

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where we assume that all matrix coefficents are real constants. The characteristic polynomial of the coefficient matrix is $p(\lambda) = \det(A - \lambda I)$. This is a polynomial degree two with real coefficients. Hence it may have two distinct roots–real or complex–or one repeated real root. In the case that the roots are distinct the coefficient matrix is diagonalizable, see Chapter 8. In the case that the root is repeated, the coefficient matrix may or may not be diagonalizable. Theorem 5.2.4 holds for a diagonalizable 2×2 coefficient matrix and it is reproduce below in the notation we use for 2×2 systems. One last statement is added to the Theorem, to address the non-diagonalizable case.

Theorem 5.3.1. *If the 2×2 constant matrix A is diagonalizable with eigenvalues $\lambda_{\pm}$ and corresponding eigenvectors $\boldsymbol{v}^{\pm}$, then the general solution to the linear system $\boldsymbol{x}' = A\,\boldsymbol{x}$ is*

$$\boldsymbol{x}_{\text{gen}}(t) = c_+ \boldsymbol{v}^+ e^{\lambda_+ t} + c_- \boldsymbol{v}^- e^{\lambda_- t}. \tag{5.3.1}$$

We classify the 2×2 linear systems by the eigenvalues of their coefficient matrix:

(A) The eigenvalues λ_+, λ_- are real and distinct;
(B) The eigenvalues $\lambda_{\pm} = \alpha \pm \beta i$ are distinct and complex, with $\lambda_+ = \overline{\lambda_-}$;
(C) The eigenvalues $\lambda_+ = \lambda_- = \lambda_0$ is repeated and real.

We now provide a few examples of systems on each of the cases above, starting with an example of case (A).

Example 5.3.1: Find the general solution of the 2×2 linear system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Solution: We have computed in Example 5.2.3 the eigenvalues and eigenvectors of the coefficient matrix,

$$\lambda_+ = 4, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This coefficient matrix has distinct real eigenvalues, so the general solution to the differential equation is

$$\boldsymbol{x}_{\text{gen}}(t) = c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

◁

We now focus on case (B). The coefficient matrix is real-valued with the complex-valued eigenvalues. In this case each eigenvalue is the complex conjugate of the other. A similar result is true for $n \times n$ real-valued matrices. When such $n \times n$ matrix has a complex eigenvalue λ, there is another eigenvalue $\overline{\lambda}$. A similar result holds for the respective eigenvectors.

Theorem 5.3.2 (Conjugate pairs). *If an $n \times n$ real-valued matrix A has a complex eigenvalue eigenvector pair λ, $\boldsymbol{v}$, then the complex conjugate pair $\overline{\lambda}$, $\overline{\boldsymbol{v}}$ is an eigenvalue eigenvector pair of matrix A.*

Proof of Theorem 5.3.2: Complex conjugate the eigenvalue eigenvector equation for λ and $\boldsymbol{v}$, and recall that matrix A is real-valued, hence $\overline{A} = A$. We obtain,

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \quad \Leftrightarrow \quad A\overline{\boldsymbol{v}} = \overline{\lambda}\, \overline{\boldsymbol{v}},$$

This establishes the Theorem. □

Complex eigenvalues of a matrix with real coefficients are always complex conjugate pairs. Same it's true for their respective eigenvectors. So they can be written in terms of their real and imaginary parts as follows,

$$\lambda_{\pm} = \alpha \pm i\beta, \qquad \boldsymbol{v}^{(\pm)} = \boldsymbol{a} \pm i\boldsymbol{b}, \tag{5.3.2}$$

where α, $\beta \in \mathbb{R}$ and $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^n$.

The general solution formula in Eq. (5.3.1) still holds in the case that A has complex eigenvalues and eigenvectors. The main drawback of this formula is similar to what we found in Chapter 2. It is difficult to separate real-valued from complex-valued solutions. The fix to that problem is also similar to the one found in Chapter 2: Find a real-valued fundamental set of solutions. The following result holds for $n \times n$ systems.

Theorem 5.3.3 (Complex and real solutions). *If $\lambda_{\pm} = \alpha \pm i\beta$ are eigenvalues of an $n \times n$ constant matrix A with eigenvectors $\boldsymbol{v}^{(\pm)} = \boldsymbol{a} \pm i\boldsymbol{b}$, where α, $\beta \in \mathbb{R}$ and $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^n$, and $n \geqslant 2$, then a linearly independent set of two complex-valued solutions to $\boldsymbol{x}' = A\,\boldsymbol{x}$ is*

$$\big\{ \boldsymbol{x}^{+}(t) = \boldsymbol{v}^{+}\, e^{\lambda_{+} t},\ \boldsymbol{x}^{-}(t) = \boldsymbol{v}^{-}\, e^{\lambda_{-} t}, \big\}. \tag{5.3.3}$$

Furthermore, a linearly independent set of two real-valued solutions to $\boldsymbol{x}' = A\,\boldsymbol{x}$ is given by

$$\big\{ \boldsymbol{x}^{1}(t) = \big(\boldsymbol{a}\cos(\beta t) - \boldsymbol{b}\sin(\beta t)\big)\, e^{\alpha t},\ \boldsymbol{x}^{2}(t) = \big(\boldsymbol{a}\sin(\beta t) + \boldsymbol{b}\cos(\beta t)\big)\, e^{\alpha t} \big\}. \tag{5.3.4}$$

Proof of Theorem 5.3.3: Theorem 5.2.4 implies the the set in (5.3.3) is a linearly independent set. The new information in Theorem 5.3.3 above is the real-valued solutions in Eq. (5.3.4). They are obtained from Eq. (5.3.3) as follows:

$$\begin{aligned}
\boldsymbol{x}^{\pm} &= (\boldsymbol{a} \pm i\boldsymbol{b})\, e^{(\alpha \pm i\beta)t} \\
&= e^{\alpha t}(\boldsymbol{a} \pm i\boldsymbol{b})\, e^{\pm i\beta t} \\
&= e^{\alpha t}(\boldsymbol{a} \pm i\boldsymbol{b})\,\big(\cos(\beta t) \pm i\sin(\beta t)\big) \\
&= e^{\alpha t}\big(\boldsymbol{a}\cos(\beta t) - \boldsymbol{b}\sin(\beta t)\big) \pm i e^{\alpha t}\big(\boldsymbol{a}\sin(\beta t) + \boldsymbol{b}\cos(\beta t)\big).
\end{aligned}$$

Since the differential equation $\boldsymbol{x}' = A\boldsymbol{x}$ is linear, the functions below are also solutions,

$$\begin{aligned}
\boldsymbol{x}^{1} &= \frac{1}{2}\big(\boldsymbol{x}^{+} + \boldsymbol{x}^{-}\big) = \big(\boldsymbol{a}\cos(\beta t) - \boldsymbol{b}\sin(\beta t)\big)\, e^{\alpha t}, \\
\boldsymbol{x}^{2} &= \frac{1}{2i}\big(\boldsymbol{x}^{+} - \boldsymbol{x}^{-}\big) = \big(\boldsymbol{a}\sin(\beta t) + \boldsymbol{b}\cos(\beta t)\big)\, e^{\alpha t}.
\end{aligned}$$

This establishes the Theorem. □

Example 5.3.2: Find a real-valued set of fundamental solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}. \tag{5.3.5}$$

Solution: Fist find the eigenvalues of matrix A above,

$$0 = \begin{vmatrix} (2-\lambda) & 3 \\ -3 & (2-\lambda) \end{vmatrix} = (\lambda - 2)^2 + 9 \quad \Rightarrow \quad \lambda_{\pm} = 2 \pm 3i.$$

Then find the respective eigenvectors. The one corresponding to λ_+ is the solution of the homogeneous linear system with coefficients given by

$$\begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \to \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}.$$

Therefore the eigenvector $\boldsymbol{v}^{+} = \begin{bmatrix} v_1^{+} \\ v_2^{+} \end{bmatrix}$ is given by

$$v_1^{+} = -iv_2^{+} \quad \Rightarrow \quad v_2^{+} = 1, \quad v_1^{+} = -i, \quad \Rightarrow \quad \boldsymbol{v}^{+} = \begin{bmatrix} -i \\ 1 \end{bmatrix}, \quad \lambda_{+} = 2+3i.$$

The second eigenvector is the complex conjugate of the eigenvector found above, that is,

$$\boldsymbol{v}^{-} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \lambda_{-} = 2-3i.$$

Notice that

$$\boldsymbol{v}^{(\pm)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 0 \end{bmatrix} i.$$

Then, the real and imaginary parts of the eigenvalues and of the eigenvectors are given by

$$\alpha = 2, \qquad \beta = 3, \qquad \boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

So a real-valued expression for a fundamental set of solutions is given by

$$\boldsymbol{x}^{1} = \Big(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(3t) - \begin{bmatrix} -1 \\ 0 \end{bmatrix} \sin(3t) \Big) e^{2t} \quad \Rightarrow \quad \boldsymbol{x}^{1} = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t},$$

$$\boldsymbol{x}^{2} = \Big(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(3t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cos(3t) \Big) e^{2t} \quad \Rightarrow \quad \boldsymbol{x}^{2} = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t}.$$

◁

We end with case (C). There are no many possibilities left for a 2×2 real matrix that both is diagonalizable and has a repeated eigenvalue. Such matrix must be proportional to the identity matrix.

Theorem 5.3.4. *Every 2×2 diagonalizable matrix with repeated eigenvalue λ_0 has the form*

$$A = \lambda_0 I.$$

Proof of Theorem 5.3.4: Since matrix A diagonalizable, there exists a matrix P invertible such that $A = PDP^{-1}$. Since A is 2×2 with a repeated eigenvalue $\lambda_{|tizero}$, then

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \lambda_0 \, I_2.$$

Put these two fatcs together,

$$A = P\lambda_0 I P^{-1} = \lambda_0 P \, P^{-1} = \lambda_0 I.$$

□

Remark: The general solution $\boldsymbol{x}_{\rm gen}$ for $\boldsymbol{x}' = \lambda I \, \boldsymbol{x}$ is simple to write. Since any non-zero 2-vector is an eigenvector of $\lambda_0 I_2$, we choos the linearly independent set

$$\Big\{ \boldsymbol{v}^{1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{v}^{2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Big\}.$$

Using these eigenvectors we can write the general solution,

$$\boldsymbol{x}_{\rm gen}(t) = c_1 \boldsymbol{v}^{1} e^{\lambda_0 t} + c_2 \boldsymbol{v}^{2} e^{\lambda_0 t} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda_0 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\lambda_0 t} \quad \Rightarrow \quad \boldsymbol{x}_{\rm gen}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}.$$

5.3.2. **Non-diagonalizable systems.** A 2×2 linear systems might not be diagonalizable. This can happen only when the coefficient matrix has a repeated eigenvalue and all eigenvectors are proportional to each other. If we denote by λ the repeated eigenvalue of a 2×2 matrix A, and by $\boldsymbol{v}$ an associated eigenvector, then one solution to the differential system $\boldsymbol{x}' = A\,\boldsymbol{x}$ is

$$\boldsymbol{x}^{1}(t) = \boldsymbol{v}\,e^{\lambda t}.$$

Every other eigenvector $\tilde{\boldsymbol{v}}$ associated with λ is proportional to $\boldsymbol{v}$. So any solution of the form $\tilde{\boldsymbol{v}}\,e^{\lambda t}$ is proportional to the solution above. The next result provides a linearly independent set of two solutions to the system $\boldsymbol{x}' = A\boldsymbol{x}$ associated with the repeated eigenvalue λ.

Theorem 5.3.5 (Repeated eigenvalue). *If an 2×2 matrix A has a repeated eigenvalue λ with only one associated eigen-direction, given by the eigenvector $\boldsymbol{v}$, then the differential system $\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$ has a linearly independent set of solutions*

$$\big\{\boldsymbol{x}^{1}(t) = \boldsymbol{v}\,e^{\lambda t}, \quad \boldsymbol{x}^{2}(t) = \big(\boldsymbol{v}\,t + \boldsymbol{w}\big)\,e^{\lambda t}\big\},$$

where the vector $\boldsymbol{w}$ is one of infinitely many solutions of the algebraic linear system

$$(A - \lambda I)\boldsymbol{w} = \boldsymbol{v}. \tag{5.3.6}$$

Remark: The eigenvalue λ is the precise number that makes matrix $(A-\lambda I)$ not invertible, that is, $\det(A-\lambda I) = 0$. This implies that an algebraic linear system with coefficient matrix $(A-\lambda I)$ is not consistent for every source. Nevertheless, the Theorem above says that Eq. (5.3.6) has solutions. The fact that the source vector in that equation is $\boldsymbol{v}$, an eigenvector of A, is crucial to show that this system is consistent.

Proof of Theorem 5.3.5: One solution to the differential system is $\boldsymbol{x}^{1}(t) = \boldsymbol{v}\,e^{\lambda t}$. Inspired by the reduction order method we look for a second solution of the form

$$\boldsymbol{x}^{2}(t) = \boldsymbol{u}(t)\,e^{\lambda t}.$$

Inserting this function into the differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ we get

$$\boldsymbol{u}' + \lambda\,\boldsymbol{u} = A\,\boldsymbol{u} \quad \Rightarrow \quad (A-\lambda I)\,\boldsymbol{u} = \boldsymbol{u}'.$$

We now introduce a power series expansion of the vector-valued function $\boldsymbol{u}$,

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \boldsymbol{u}_1 t + \boldsymbol{u}_2 t^2 + \cdots,$$

into the differential equation above,

$$(A-\lambda I)(\boldsymbol{u}_0 + \boldsymbol{u}_1 t + \boldsymbol{u}_2 t^2 + \cdots) = (\boldsymbol{u}_1 + 2\boldsymbol{u}_2 t + \cdots).$$

If we evaluate the equation above at $t = 0$, and then its derivative at $t = 0$, and so on, we get the following infinite set of linear algebraic equations

$$\begin{aligned} (A-\lambda I)\boldsymbol{u}_0 &= \boldsymbol{u}_1,\\ (A-\lambda I)\boldsymbol{u}_1 &= 2\boldsymbol{u}_2,\\ (A-\lambda I)\boldsymbol{u}_2 &= 3\boldsymbol{u}_3\\ &\vdots \end{aligned}$$

Here is where we use Cayley-Hamilton's Theorem. Recall that the characteristic polynomial $p(\tilde{\lambda}) = \det(A - \tilde{\lambda} I)$ has the form

$$p(\tilde{\lambda}) = \tilde{\lambda}^2 - \operatorname{tr}(A)\,\tilde{\lambda} + \det(A).$$

Cayley-Hamilton Theorem says that the matrix-valued polynomial $p(A) = 0$, that is,

$$A^2 - \operatorname{tr}(A)\,A + \det(A)\,I = 0.$$

Since in the case we are interested in matrix A has a repeated root λ, then

$$p(\tilde\lambda) = (\tilde\lambda - \lambda)^2 = \tilde\lambda^2 - 2\lambda\,\tilde\lambda + \lambda^2.$$

Therefore, Cayley-Hamilton Theorem for the matrix in this Theorem has the form

$$0 = A^2 - 2\lambda\, A + \lambda^2\, I \quad \Rightarrow \quad (A - \lambda I)^2 = 0.$$

This last equation is the one we need to solve the system for the vector-valued $\boldsymbol{u}$. Multiply the first equation in the system by $(A - \lambda I)$ and use that $(A - \lambda I)^2 = 0$, then we get

$$\boldsymbol{0} = (A - \lambda I)^2 \boldsymbol{u}_0 = (A - \lambda I)\, \boldsymbol{u}_1 \quad \Rightarrow \quad (A - \lambda I)\boldsymbol{u}_1 = \boldsymbol{0}.$$

This implies that $\boldsymbol{u}_1$ is an eigenvector of A with eigenvalue λ. We can denote it as $\boldsymbol{u}_1 = \boldsymbol{v}$. Using this information in the rest of the system we get

$$\begin{aligned} (A - \lambda I)\boldsymbol{u}_0 &= \boldsymbol{v}, \\ (A - \lambda I)\boldsymbol{v} &= 2\boldsymbol{u}_2 \quad \Rightarrow \quad \boldsymbol{u}_2 = \boldsymbol{0}, \\ (A - \lambda I)\boldsymbol{u}_2 &= 3\boldsymbol{u}_3 \quad \Rightarrow \quad \boldsymbol{u}_3 = \boldsymbol{0}, \\ &\vdots \end{aligned}$$

We conclude that all terms $\boldsymbol{u}_2 = \boldsymbol{u}_3 = \cdots = \boldsymbol{0}$. Denoting $\boldsymbol{u}_0 = \boldsymbol{w}$ we obtain the following system of algebraic equations,

$$\begin{aligned} (A - \lambda I)\boldsymbol{w} &= \boldsymbol{v}, \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}. \end{aligned}$$

For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ solution of the system above we get $\boldsymbol{u}(t) = \boldsymbol{w} + t\boldsymbol{v}$. This means that the second solution to the differential equation is

$$\boldsymbol{x}^2(t) = (t\boldsymbol{v} + \boldsymbol{w})\, e^{\lambda t}.$$

This establishes the Theorem. □

EXAMPLE 5.3.3: Find the fundamental solutions of the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \frac{1}{4}\begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

SOLUTION: As usual, we start finding the eigenvalues and eigenvectors of matrix A. The former are the solutions of the characteristic equation

$$0 = \begin{vmatrix} \left(-\frac{3}{2} - \lambda\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} - \lambda\right) \end{vmatrix} = \Big(\lambda + \frac{3}{2}\Big)\Big(\lambda + \frac{1}{2}\Big) + \frac{1}{4} = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2.$$

Therefore, there solution is the repeated eigenvalue $\lambda = -1$. The associated eigenvectors are the vectors $\boldsymbol{v}$ solution to the linear system $(A + I)\boldsymbol{v} = \boldsymbol{0}$,

$$\begin{bmatrix} \left(-\frac{3}{2} + 1\right) & 1 \\ -\frac{1}{4} & \left(-\frac{1}{2} + 1\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 2v_2.$$

Choosing $v_2 = 1$, then $v_1 = 2$, and we obtain

$$\lambda = -1, \qquad \boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Any other eigenvector associated to $\lambda = -1$ is proportional to the eigenvector above. The matrix A above is not diagonalizable. So. we follow Theorem 5.3.5 and we solve for a vector $\boldsymbol{w}$ the linear system $(A + I)\boldsymbol{w} = \boldsymbol{v}$. The augmented matrix for this system is given by,

$$\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 1 & -2 & -4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad w_1 = 2w_2 - 4.$$

We have obtained infinitely many solutions given by

$$\boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

As one could have imagined, given any solution $\boldsymbol{w}$, the $c\boldsymbol{v} + \boldsymbol{w}$ is also a solution for any $c \in \mathbb{R}$. We choose the simplest solution given by

$$\boldsymbol{w} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}.$$

Therefore, a fundamental set of solutions to the differential equation above is formed by

$$\boldsymbol{x}^{1}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}, \qquad \boldsymbol{x}^{2}(t) = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) e^{-t}. \tag{5.3.7}$$

$\lhd$

5.3.3. **Exercises.**

5.3.1.- .

5.3.2.- .

5.4. Two-by-Two Phase Portraits

Figures are easier to understand than words. Words are easier to understand than equations. The qualitative behavior of a function is often simpler to visualize from a graph than from an explicit or implicit expression of the function.

Take, for example, the differential equation

$$y'(t) = \sin(y(t)).$$

This equation is separable and the solution can be obtained using the techniques in Section 1.3. They lead to the following implicit expression for the solution y,

$$-\ln\big|\csc(y) + \cot(y)\big| = t + c.$$

Although this is an exact expression for the solution of the differential equation, the qualitative behavior of the solution is not so simple to understand from this formula. The graph of the solution, however, given on the right, provides us with a clear picture of the solution behavior. In this particular case the graph of the solution can be computed from the equation itself, without the need to solve the equation.

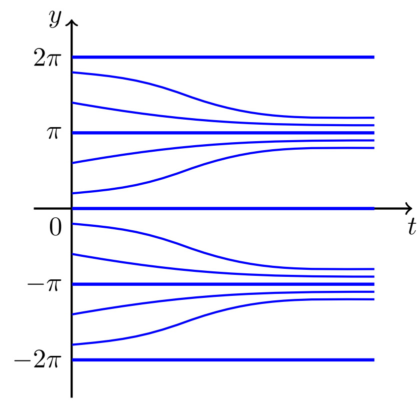


Figure 25. Several solutions of the equation $y' = \sin(y)$

In the case of 2×2 systems the solution vector has the form

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Two functions define the solution vector. In this case one usually graphs each component of the solution vector, x_1 and x_2, as functions of t. There is, however, another way to graph a 2-vector-valued function: plot the whole vector $\boldsymbol{x}(t)$ at t on the plane x_1, x_2. Each vector $\boldsymbol{x}(t)$ is represented by its end point, while the whole solution $\boldsymbol{x}$ represents a line with arrows pointing in the direction of increasing t. Such a figure is called a *phase diagram* or *phase portrait*.

In the case that the solution vector $\boldsymbol{x}(t)$ is interpreted as the position function of a particle moving in a plane at the time t, the curve given in the phase portrait is the trajectory of the particle. The arrows added to this trajectory indicate the motion of the particle as time increases.

In this Section we say how to plot phase portraits. We focus on solutions to the systems studied in the previous Section 5.3–2×2 homogeneous, constant coefficient linear systems

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t). \tag{5.4.1}$$

Theorem 5.3.1 spells out the general solution in the case the coefficient matrix is diagonalizable with eigenpairs $\lambda_\pm$, $\boldsymbol{v}^\pm$. The general solution is given by

$$\boldsymbol{x}_{\text{gen}}(t) = c_+ \boldsymbol{v}^+ e^{\lambda_+ t} + c_- \boldsymbol{v}^- e^{\lambda_- t}. \tag{5.4.2}$$

Solutions with real distinct eigenvalues are essentially different from solutions with complex eigenvalues. Those differences can be seen in their phase portraits. Both solution types are essentially different from solutions with a repeated eigenvalue. We now study each case.

5.4.1. **Real distinct eigenvalues.** We study the system in (5.4.1) in the case that matrix A has two real eigenvalues $\lambda_+ \neq \lambda_-$. The case where one eigenvalues vanishes is left one of the exercises at the end of the Section. We study the case where both eigenvalues are non-zero. Two non-zero eigenvalues belong to one of hte following cases:

(i) $\lambda_+ > \lambda_- > 0$, both eigenvalues positive;
(ii) $\lambda_+ > 0 > \lambda_-$, one eigenvalue negative and the other positive;
(iii) $0 > \lambda_+ > \lambda_-$, both eigenvalues negative.

In a phase portrait the solution vector $\boldsymbol{x}(t)$ at t is displayed on the plane x_1, x_2. The whole vector is shown, only the end point of the vector is shown for $t \in (-\infty, \infty)$. The result is a curve in the x_1, x_2 plane. One usually adds arrows to determine the direction of increasing t. A phase portrait contains several curves, each one corresponding to a solution given in Eq. (5.4.2) for particular choice of constants c_+ and c_-. A phase diagram can be sketched by following these few steps:

(a) Plot the eigenvectors $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ corresponding to the eigenvalues λ_+ and λ_-.
(b) Draw the whole lines parallel to these vectors and passing through the origin. These straight lines correspond to solutions with either c_+ or c_- zero.
(c) Draw arrows on these lines to indicate how the solution changes as the variable t increases. If t is interpreted as time, the arrows indicate how the solution changes into the future. The arrows point towards the origin if the corresponding eigenvalue λ is negative, and they point away form the origin if the eigenvalue is positive.
(d) Find the non-straight curves correspond to solutions with both coefficient c_+ and c_- non-zero. Again, arrows on these curves indicate the how the solution moves into the future.

Case $\lambda_+ > \lambda_- > 0$.

EXAMPLE 5.4.1: Sketch the phase diagram of the solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \frac{1}{4}\begin{bmatrix} 11 & 3 \\ 1 & 9 \end{bmatrix}. \tag{5.4.3}$$

SOLUTION: The characteristic equation for this matrix A is given by

$$\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_+ = 3, \\ \lambda_- = 2. \end{cases}$$

One can show that the corresponding eigenvectors are given by

$$\boldsymbol{v}^+ = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}^- = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

So the general solution to the differential equation above is given by

$$\boldsymbol{x}(t) = c_+ \, \boldsymbol{v}^+ e^{\lambda_+ t} + c_- \, \boldsymbol{v}^- e^{\lambda_- t} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = c_+ \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{3t} + c_- \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{2t}.$$

In Fig. 26 we have sketched four curves, each representing a solution $\boldsymbol{x}$ corresponding to a particular choice of the constants c_+ and c_-. These curves actually represent eight different solutions, for eight different choices of the constants c_+ and c_-, as is described below. The arrows on these curves represent the change in the solution as the variable t grows. Since both eigenvalues are positive, the length of the solution vector always increases as t increases. The straight lines correspond to the following four solutions:

$$c_+ = 1, \; c_- = 0; \quad c_+ = 0, \; c_- = 1; \quad c_+ = -1, \; c_- = 0; \quad c_+ = 0, \; c_- = -1.$$

The curved lines on each quadrant correspond to the following four solutions:

$$c_+ = 1,\ c_- = 1; \quad c_+ = 1,\ c_- = -1; \quad c_+ = -1,\ c_- = 1; \quad c_+ = -1,\ c_- = -1.$$

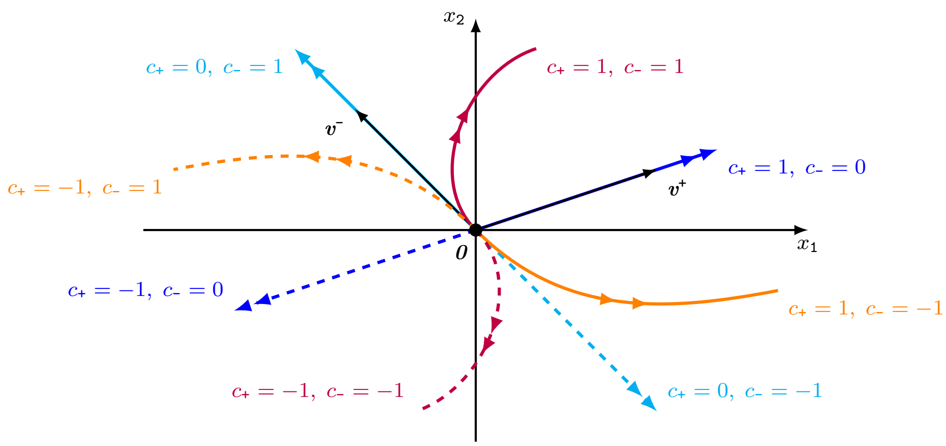


Figure 26. Eight solutions to Eq. (5.4.3), where $\lambda_+ > \lambda_- > 0$. The trivial solution $\boldsymbol{x} = \boldsymbol{0}$ is called an **unstable point**.

◁

Case $\lambda_+ > 0 > \lambda_-$.

Example 5.4.2: Sketch the phase diagram of the solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}. \tag{5.4.4}$$

Solution: In Example 5.2.3 we computed the eigenvalues and eigenvectors of the coefficient matrix, and the result was

$$\lambda_+ = 4, \quad \boldsymbol{v}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \lambda_- = -2, \quad \boldsymbol{v}^- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

In that Example we also computed the general solution to the differential equation above,

$$\boldsymbol{x}(t) = c_+ \boldsymbol{v}^+ e^{\lambda_+ t} + c_- \boldsymbol{v}^- e^{\lambda_- t} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = c_+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_- \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

In Fig. 27 we have sketched four curves, each representing a solution $\boldsymbol{x}$ corresponding to a particular choice of the constants c_+ and c_-. These curves actually represent eight different solutions, for eight different choices of the constants c_+ and c_-, as is described below. The arrows on these curves represent the change in the solution as the variable t grows. The part of the solution with positive eigenvalue increases exponentially when t grows, while the part of the solution with negative eigenvalue decreases exponentially when t grows. The straight lines correspond to the following four solutions:

$$c_+ = 1,\ c_- = 0; \quad c_+ = 0,\ c_- = 1; \quad c_+ = -1,\ c_- = 0; \quad c_+ = 0,\ c_- = -1.$$

The curved lines on each quadrant correspond to the following four solutions:

$$c_+ = 1,\ c_- = 1; \quad c_+ = 1,\ c_- = -1; \quad c_+ = -1,\ c_- = 1; \quad c_+ = -1,\ c_- = -1.$$

◁

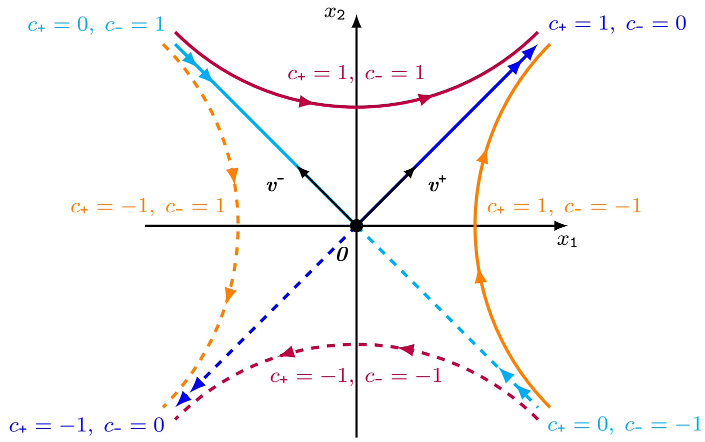


FIGURE 27. Several solutions to Eq. (5.4.4), $\lambda_+ > 0 > \lambda_-$. The trivial solution $\boldsymbol{x} = \boldsymbol{0}$ is called a **saddle point**.

Case $0 > \lambda_+ > \lambda_-$**.**

EXAMPLE 5.4.3: Sketch the phase diagram of the solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \frac{1}{4}\begin{bmatrix} -9 & 3 \\ 1 & -11 \end{bmatrix}. \tag{5.4.5}$$

SOLUTION: The characteristic equation for this matrix A is given by

$$\det(A - \lambda I) = \lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad \begin{cases} \lambda_+ = -2, \\ \lambda_- = -3. \end{cases}$$

One can show that the corresponding eigenvectors are given by

$$\boldsymbol{v}^+ = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}^- = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

So the general solution to the differential equation above is given by

$$\boldsymbol{x}(t) = c_+\, \boldsymbol{v}^+ e^{\lambda_+ t} + c_-\, \boldsymbol{v}^- e^{\lambda_- t} \quad \Leftrightarrow \quad \boldsymbol{x}(t) = c_+ \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-2t} + c_- \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-3t}.$$

In Fig. 28 we have sketched four curves, each representing a solution $\boldsymbol{x}$ corresponding to a particular choice of the constants c_+ and c_-. These curves actually represent eight different solutions, for eight different choices of the constants c_+ and c_-, as is described below. The arrows on these curves represent the change in the solution as the variable t grows. Since both eigenvalues are negative, the length of the solution vector always decreases as t grows and the solution vector always approaches zero. The straight lines correspond to the following four solutions:

$$c_+ = 1,\ c_- = 0; \quad c_+ = 0,\ c_- = 1; \quad c_+ = -1,\ c_- = 0; \quad c_+ = 0,\ c_- = -1.$$

The curved lines on each quadrant correspond to the following four solutions:

$$c_+ = 1,\ c_- = 1; \quad c_+ = 1,\ c_- = -1; \quad c_+ = -1,\ c_- = 1; \quad c_+ = -1,\ c_- = -1.$$

$\triangleleft$

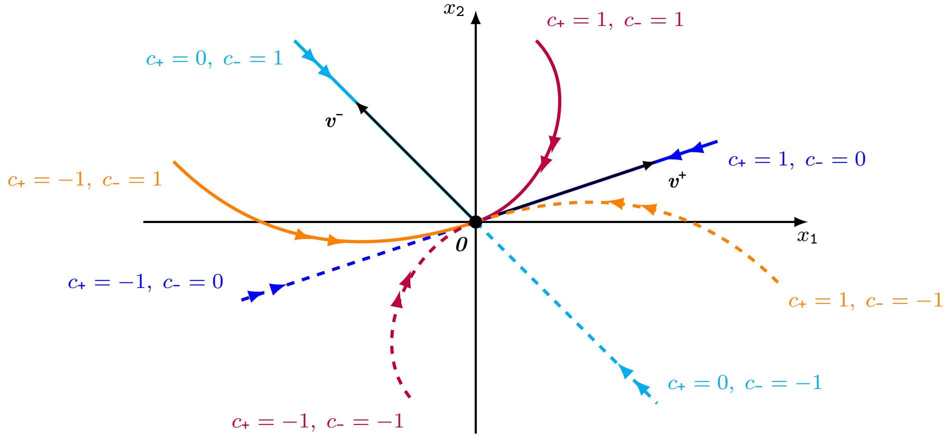


Figure 28. Several solutions to Eq. (5.4.5), where $0 > \lambda_+ > \lambda_-$. The trivial solution $\boldsymbol{x} = \boldsymbol{0}$ is called a **stable point**.

5.4.2. **Complex eigenvalues.** A real-valued matrix may have complex-valued eigenvalues. These complex eigenvalues come in pairs, because the matrix is real-valued. If λ is one of these complex eigenvalues, then $\overline{\lambda}$ is also an eigenvalue. A usual notation is $\lambda_\pm = \alpha \pm i\beta$, with $\alpha, \beta \in \mathbb{R}$. The same happens with their eigenvectors, which are written as $\boldsymbol{v}^\pm = \boldsymbol{a} \pm i\boldsymbol{b}$, with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, in the case of an $n \times n$ matrix. When the matrix is the coefficient matrix of a differential equation,

$$\boldsymbol{x}' = A\,\boldsymbol{x},$$

the solutions $\boldsymbol{x}^+(t) = \boldsymbol{v}^+ e^{\lambda_+ t}$ and $\boldsymbol{x}^-(t) = \boldsymbol{v}^- e^{\lambda_- t}$ are complex-valued. In the previous Section we presented Theorem 5.3.3, which provided real-valued solutions for the differential equation. They are the real part and the imaginary part of the solution $\boldsymbol{x}^+$, given by

$$\boldsymbol{x}^1(t) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\,e^{\alpha t}, \qquad \boldsymbol{x}^2(t) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\,e^{\alpha t}. \tag{5.4.6}$$

These real-valued solutions are used to draw phase portraits. We start with an example.

Example 5.4.4: Find a real-valued set of fundamental solutions to the differential equation below and sketch a phase portrait, where

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

Solution: We have found in Example 5.3.2 that the eigenvalues and eigenvectors of the coefficient matrix are

$$\lambda_\pm = 2 \pm 3i, \quad \boldsymbol{v}^\pm = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}.$$

Writing them in real and imaginary parts, $\lambda_\pm = \alpha \pm i\beta$ and $\boldsymbol{v}^\pm = \boldsymbol{a} \pm i\boldsymbol{b}$, we get

$$\alpha = 2, \qquad \beta = 3, \qquad \boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

These eigenvalues and eigenvectors imply the following real-valued fundamental solutions,

$$\Big\{ \boldsymbol{x}^1(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{2t}, \; \boldsymbol{x}^2(t) = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{2t} \Big\}. \tag{5.4.7}$$

The phase diagram of these two fundamental solutions is given in Fig. 29 below. There is also a circle given in that diagram, corresponding to the trajectory of the vectors

$$\tilde{\boldsymbol{x}}^1(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} \qquad \tilde{\boldsymbol{x}}^2(t) = \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix}.$$

The phase portrait of these functions is a circle, since they are unit vector-valued functions– they have length one. ◁

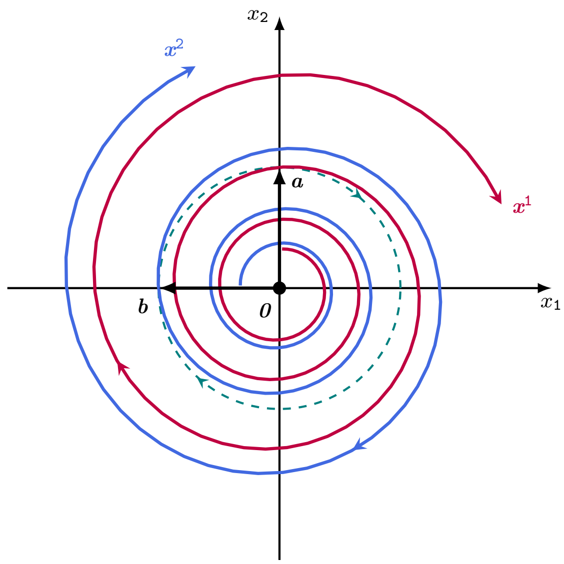


FIGURE 29. The graph of the fundamental solutions $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ in Eq. (5.4.7).

Suppose that the coefficient matris of a 2×2 differential equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ has complex eigenvalues and eigenvectors

$$\lambda_{\pm} = \alpha \pm i\beta, \qquad \boldsymbol{v}^{\pm} = \boldsymbol{a} \pm i\,\boldsymbol{b}.$$

We have said that real-valued fundamental solutions are given by

$$\boldsymbol{x}^1(t) = \big(\boldsymbol{a}\,\cos(\beta t) - \boldsymbol{b}\,\sin(\beta t)\big)\,e^{\alpha t}, \qquad \boldsymbol{x}^2(t) = \big(\boldsymbol{a}\,\sin(\beta t) + \boldsymbol{b}\,\cos(\beta t)\big)\,e^{\alpha t}.$$

We now sketch phase portraits of these solutions for a few choices of α, $\boldsymbol{a}$ and $\boldsymbol{b}$. We start fixing the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and plotting phase diagrams for solutions having $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The result can be seen in Fig. 30. For $\alpha > 0$ the solutions spiral outward as t increases, and for $\alpha < 0$ the solutions spiral inwards to the origin as t increases. The rotation direction is from vector $\boldsymbol{b}$ towards vector $\boldsymbol{a}$. The solution vector $\boldsymbol{0}$, is called unstable for $\alpha > 0$ and stable for $\alpha < 0$.

We now change the direction of vector $\boldsymbol{b}$, and we repeat the three phase portraits given above; for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The result is given in Fig. 31. Comparing Figs. 30 and 31 shows that the relative directions of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ determines the rotation direction of the solutions as t increases.

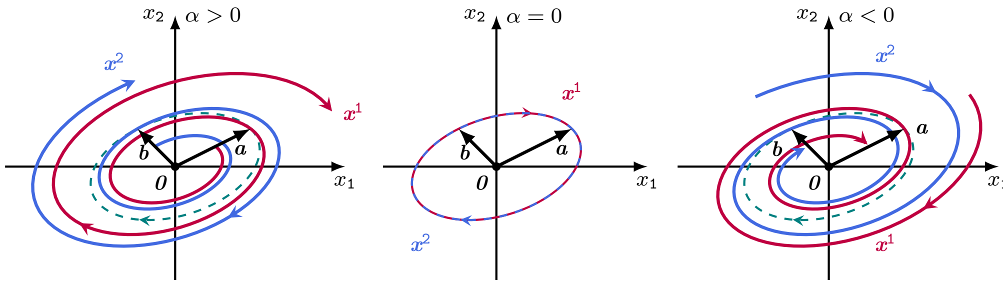


FIGURE 30. Fundamental solutions $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in Eq. (5.4.6) for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The relative positions of $\boldsymbol{a}$ and $\boldsymbol{b}$ determines the rotation direction. Compare with Fig. 31.

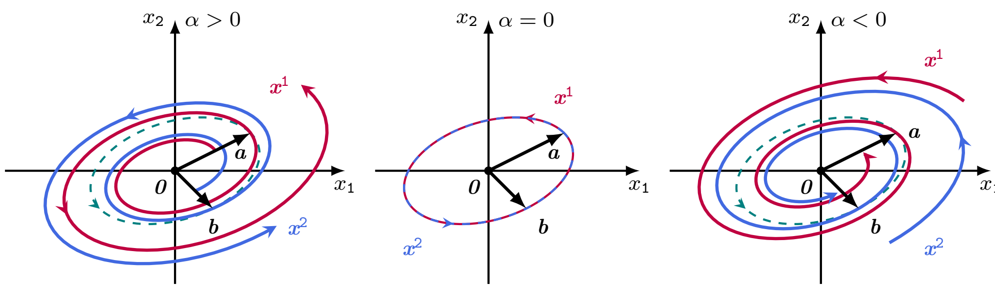


FIGURE 31. Fundamental solutions $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ in Eq. (5.4.6) for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$. The relative positions of $\boldsymbol{a}$ and $\boldsymbol{b}$ determines the rotation direction. Compare with Fig. 30.

5.4.3. **Repeated eigenvalues.** A matrix with repeated eigenvalues may or may not be diagonalizable. If a 2×2 matrix A is diagonalizable with repeated eigenvalues, then by Theorem 5.3.4 this matrix is proportional to the identity matrix, $A = \lambda_0 I$, with λ_0 the repeated eigenvalue. We saw in Section 5.3 that the general solution of a differential system with such coefficient matrix is

$$\boldsymbol{x}_{\text{gen}}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda_0 t}.$$

Phase portraits of these solutions are just straight lines, starting from the origin for $\lambda_0 > 0$, or ending at the origin for $\lambda_0 < 0$.

Non-diagonalizable 2×2 differential systems are more interesting. If $\boldsymbol{x}' = A\,\boldsymbol{x}$ is such a system, it has fundamental solutions

$$\boldsymbol{x}^1(t) = \boldsymbol{v}\, e^{\lambda_0 t}, \qquad \boldsymbol{x}^2(t) = (\boldsymbol{v}\, t + \boldsymbol{w})\, e^{\lambda_0 t}, \tag{5.4.8}$$

where λ_0 is the repeated eigenvalue of A with eigenvector $\boldsymbol{v}$, and vector $\boldsymbol{w}$ is any solution of the linear algebraic system

$$(A - \lambda_0 I)\boldsymbol{w} = \boldsymbol{v}.$$

The phase portrait of these fundamental solutions is given in Fig 32. To construct this figure start drawing the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$. The solution $\boldsymbol{x}^1$ is simpler to draw than $\boldsymbol{x}^2$, since the former is a straight semi-line starting at the origin and parallel to $\boldsymbol{v}$.

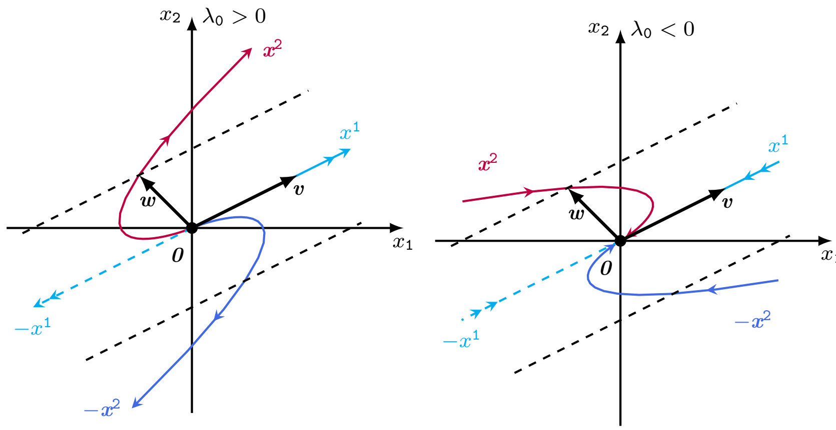


FIGURE 32. Functions $\boldsymbol{x}^1$, $\boldsymbol{x}^2$ in Eq. (5.4.8) for the cases $\lambda_0 > 0$ and $\lambda_0 < 0$.

The solution $\boldsymbol{x}^2$ is more difficult to draw. One way is to first draw the trajectory of the time-dependent vector

$$\tilde{\boldsymbol{x}}^2 = \boldsymbol{v}\,t + \boldsymbol{w}.$$

This is a straight line parallel to $\boldsymbol{v}$ passing through $\boldsymbol{w}$, one of the black dashed lines in Fig. 32, the one passing through $\boldsymbol{w}$. The solution $\boldsymbol{x}^2$ differs from $\tilde{\boldsymbol{x}}^2$ by the multiplicative factor $e^{\lambda_0 t}$. Consider the case $\lambda_0 > 0$. For $t > 0$ we have $\boldsymbol{x}^2(t) > \tilde{\boldsymbol{x}}^2(t)$, and the opposite happens for $t < 0$. In the limit $t \to -\infty$ the solution values $\boldsymbol{x}^2(t)$ approach the origin, since the exponential factor $e^{\lambda_0 t}$ decreases faster than the linear factor t increases. The result is the purple line in the first picture of Fig. 32. The other picture, for $\lambda_0 < 0$ can be constructed following similar ideas.

5.4.4. **Exercises.**

5.4.1.- .

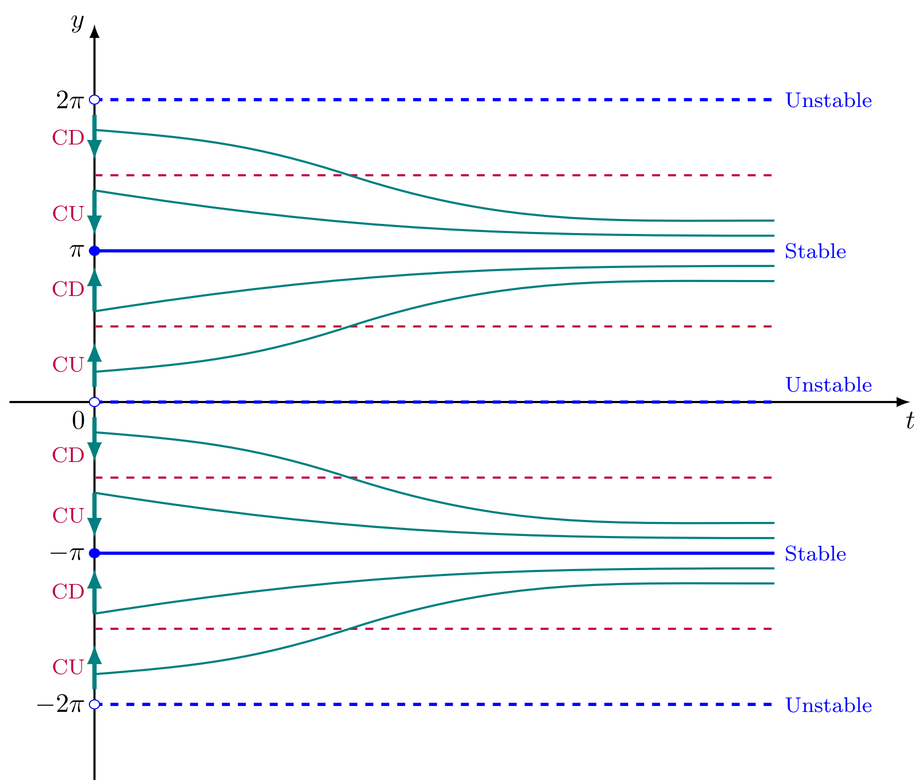
5.4.2.- .

5.5. Non-Diagonalizable Systems

Coming up.

Chapter 6. Autonomous Systems and Stability

By the end of the seventeenth century Newton had invented differential equations, discovered his laws of motion and the law of universal gravitation. He combined all of them to explain Kepler laws of planetary motion. Newton solved what now is called the two-body problem. Kepler laws correspond to the case of one planet orbiting the Sun. People then started to study the three-body problem. For example the movement of Earth, Moon, and Sun. This problem turned out to be far more difficult than the two-body problem and no solution was ever found. Around the end of the nineteenth century Henri Poincaré proved a breakthrough result. The solutions of the three body problem could not be found explicitly in terms of elementary functions, such as combinations of polynomials, trigonometric functions, exponential, and logarithms. This led him to invent the so-called Qualitative Theory of Differential Equations. In this theory one studies the geometric properties of solutions–whether they show periodic behavior, tend to fixed points, tend to infinity, etc. This approach evolved into the modern field of dynamics. In this chapter we introduce a few basic concepts and we use them to find qualitative information of a particular type of differential equations, called autonomous equations.



6.1. Flows on the Line

This whole chapter is dedicated to study the qualitative behavior of solutions to differential equations without actually computing the explicit expression of these solutions. In this section we concentrate on first order differential equations in one unknown function. We already have studied these equations in Chapter 1, § 1.1-1.4, and we have found formulas for their solutions. In this section we use these equations to present a new method to study qualitative properties of their solutions. Knowing the exact solution to the equation will help us understand how this new method works. In the next section we generalize this method to systems of two equations for two unknown functions.

6.1.1. Autonomous Equations.

In this section we study, one more time, first order nonlinear differential equations. In § 1.3 we learned how to solve these equations. We integrated on both sides of the equation. We then got an implicit expression for the solution in terms of the antiderivative of the equation coefficients. In this section we concentrate on a particular type of separable equations, called autonomous, where the independent variable does not appear explicitly in the equation. For these systems we find a few qualitative properties of their solutions without actually computing the solution. We find these properties of the solutions by studying the equation itself.

Definition 6.1.1. *An* ***autonomous equation*** *is a first order differential equation for the unknown function y and independent variable t given by*

$$y' = f(y), \tag{6.1.1}$$

that is the independent variable t does not appear explicitly in the equation.

Remarks: The equation in (6.1.1) is separable, since it has the form

$$h(y)\, y' = g(t),$$

as in Def. 1.3.1, with $h(y) = 1/f(y)$ and $g(t) = 1$.

The autonomous equations we study in this section are a particular type of the separable equations we studied in § 1.3, as we can see in the following examples.

Example 6.1.1: The following first order separable equations are autonomous:

(a) $y' = 2\,y + 3$.

(b) $y' = \sin(y)$.

(c) $y' = r\,y\left(1 - \dfrac{y}{K}\right)$.

The independent variable t does not appear explicitly in these equations. The following equations are not autonomous.

(a) $y' = 2\,y + 3t$.

(b) $y' = t^2 \sin(y)$.

(c) $y' = t\,y\left(1 - \dfrac{y}{K}\right)$. ◁

Sometimes an autonomous equation is simple to solve, explicitly. Even more, the solutions are simple to understand. For example the graph of the solution is simple to do. Here is a well known example.

Example 6.1.2: Find all solutions of the first order autonomous system

$$y' = a\,y + b, \qquad a,\, b > 0.$$

Solution:

This is a linear, constant coefficients equation, so it could be solved using the integrating factor method. But this is also a separable equation, so we solve it as follows,

$$\int \frac{dy}{a\,y+b} = \int dt \;\Rightarrow\; \frac{1}{a}\,\ln(a\,y+b) = t + c_0$$

so we get,

$$a\,y + b = e^{at} e^{ac_0}$$

and denoting $c = e^{ac_0}/a$, we get the expression

$$y(t) = c\,e^{at} - \frac{b}{a}. \tag{6.1.2}$$

This is the expression for the solution we got in Theorem 1.1.2. ◁

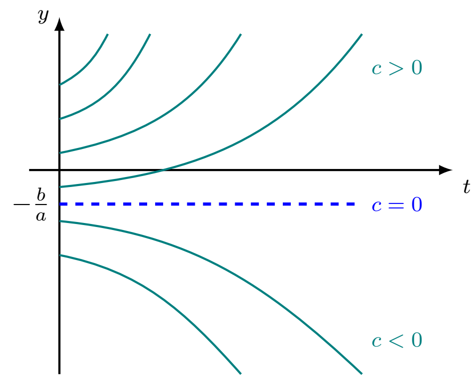


Figure 33. A few solutions to Eq. (6.1.2) for different c.

However, the solutions of an autonomous equation are sometimes not so simple to understand. Even in the case that we can solve the differential equation.

Example 6.1.3: Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0) = y_0$.

Solution: We first find the exact solutions and then we see if we can graph them. The equation is separable, then

$$\frac{y'(t)}{\sin\big(y(t)\big)} = 1 \quad \Rightarrow \quad \int_0^t \frac{y'(t)}{\sin\big(y(t)\big)}\,dt = t.$$

Use the usual substitution $u = y(t)$, so $du = y'(t)\,dt$, so we get

$$\int_{y_0}^{y(t)} \frac{du}{\sin(u)} = t.$$

In an integration table we can find that

$$\ln\Big[\frac{\sin(u)}{1+\cos(u)}\Big]\Big|_{y_0}^{y(t)} = t \quad \Rightarrow \quad \ln\Big[\frac{\sin(y)}{1+\cos(y)}\Big] - \ln\Big[\frac{\sin(y_0)}{1+\cos(y_0)}\Big] = t.$$

We can rewrite the expression above in terms of one single logarithm,

$$\ln\Big[\frac{\sin(y)}{\big(1+\cos(y)\big)}\,\frac{\big(1+\cos(y_0)\big)}{\sin(y_0)}\Big] = t.$$

If we compute the exponential on both sides of the equation above we get an implicit expression of the solution,

$$\frac{\sin(y)}{\big(1+\cos(y)\big)} = \frac{\sin(y_0)}{\big(1+\cos(y_0)\big)}\,e^{t}. \tag{6.1.3}$$

Although we have the solution, in this case in implicit form, it is not simple to graph that solution without the help of a computer. So, we do not sketch the graph right now. ◁

Sometimes the exact expression for the solution of a differential equation is difficult to interpret. For example, take the solution in (6.1.3), in Example 6.1.3. It is not so easy to see, for an arbitrary initial condition y_0, what is the behavior of the solution values $y(t)$ as $t \to \infty$. To be able to answer questions like this one is that we introduce a new approach, a geometric approach.

6.1.2. **A Geometric Analysis.** The idea is to obtain qualitative information about solutions to an autonomous equation using the equation itself, without solving it. We now use the equation of Example 6.1.3 to show how this can be done.

EXAMPLE 6.1.4: Sketch a qualitative graph of solutions to $y' = \sin(y)$, for different initial data conditions $y(0)$.

SOLUTION: The differential equation has the form $y' = f(y)$, where $f(y) = \sin(y)$. The first step in the graphical approach is to graph the function f.

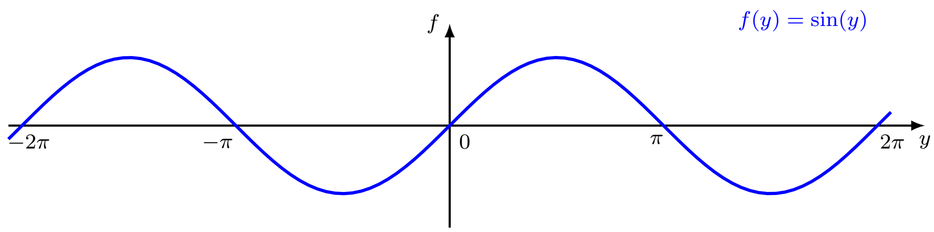


FIGURE 34. Graph of the function $f(y) = \sin(y)$.

The second step is to identify all the zeros of the function f. In this case,

$$f(y) = \sin(y) = 0 \quad \Rightarrow \quad y_n = n\pi, \quad \text{where} \quad n = \cdots, -2, -1, 0, 1, 2, \cdots .$$

It is important to realize that these constants y_n are solutions of the differential equation. On the one hand, they are constants, t-independent, so $y_n' = 0$. On the other hand, these constants y_n are zeros of f, hence $f(y_n) = 0$. So y_n are solutions of the differential equation

$$0 = y_n' = f(y_n) = 0.$$

These t-independent solutions, y_n, are called *stationary solutions*. They are also called equilibrium solutions, or fixed points, or critical points.

The third step is to identify the regions on the line where f is positive, and where f is negative. These regions are bounded by the critical points. Now, in an interval where $f > 0$ write a right arrow, and in the intervals where $f < 0$ write a left arrow, as shown below.

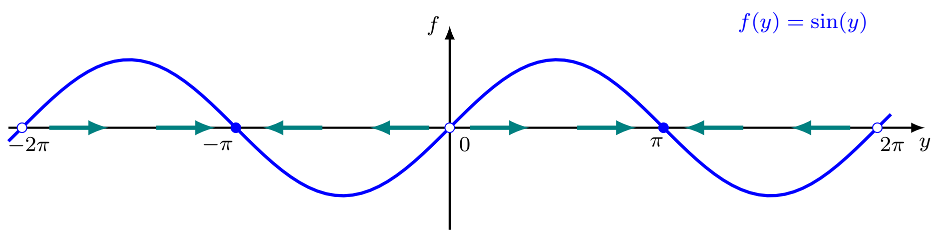


FIGURE 35. Critical points and increase/decrease information added to Fig. 34.

It is important to notice that in the regions where $f > 0$ a solution y is increasing. And in the regions where $f < 0$ a solution y is decreasing. The reason for this claim is, of course, the differential equation, $f(y) = y'$.

The fourth step is to find the regions where the curvature of a solution is concave up or concave down. That information is given by y''. But the differential equation relates y'' to

$f(y)$ and $f'(y)$. By the chain rule,

$$y'' = \frac{d}{dt}\Big(\frac{dy}{dt}\Big) = \frac{d}{dt} f(y(t)) = \frac{df}{dy}\,\frac{dy}{dt} \quad \Rightarrow \quad y'' = f'(y)\, f(y)$$

So the regions where $f(y)\, f'(y) > 0$ a solution is concave up (CU), and the regions where $f(y)\, f'(y) < 0$ a solution is concave down (CD).

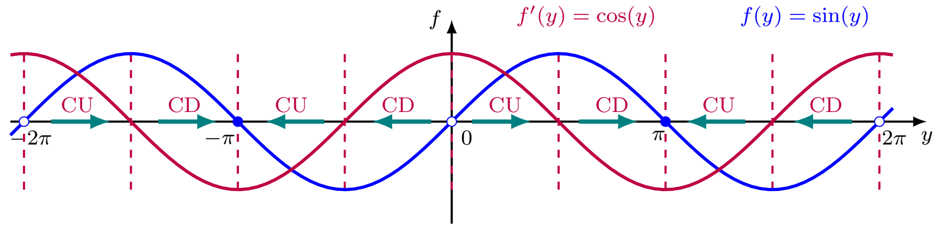


Figure 36. Concavity information on the solution y added to Fig. 35.

This is all the information we need to sketch a qualitative graph of solutions to the differential equation. So, the last step is to put all this information on a yt-plane. The horizontal axis above is now the vertical axis, and we now plot soltuions y of the differential equation. The result is given below.

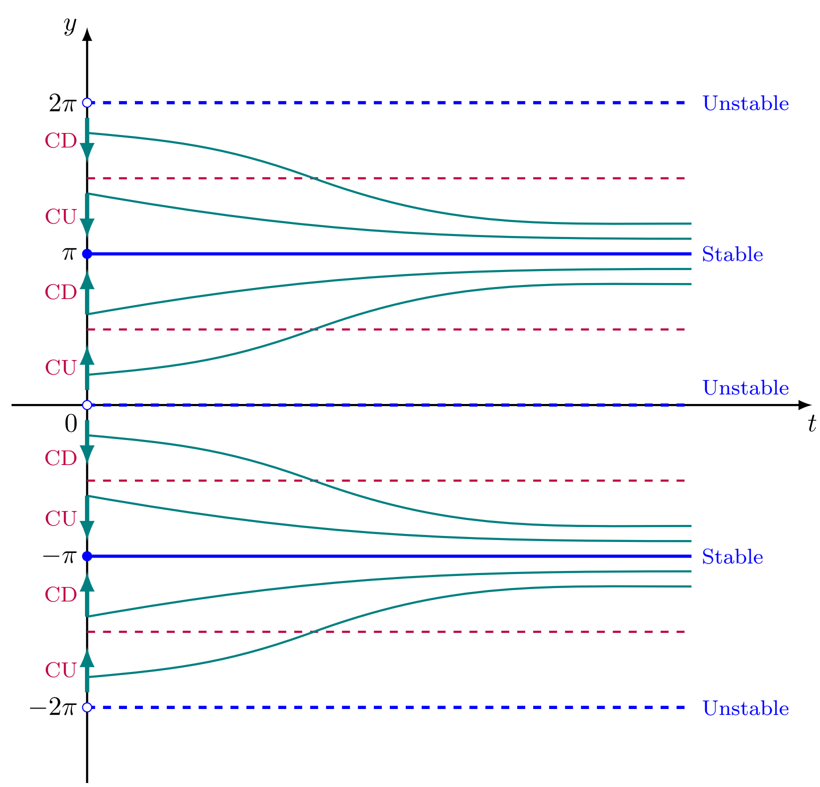


Figure 37. Qualitative graphs of solutions y for different initial conditions.

The picture above contains the graph of several solutions y for different choices of initial data $y(0)$. Stationary solutions are in blue, t-dependent solutions in green. The stationary solutions are separated in two types. The stable solutions $y_{-1} = -\pi$, $y_1 = \pi$, are pictured with solid blue lines. The unstable solutions $y_{-2} = -2\pi$, $y_0 = 0$, $y_2 = 2\pi$, are pictured with dashed blue lines. ◁

Remark: A qualitative graph of the solutions does not provide all the possible information about the solution. For example, we know from the graph above that for some initial conditions the corresponding solutions have inflection points at some $t > 0$. But we cannot know the exact value of t where the inflection point occurs. Such information could be useful to have, since $|y'|$ has its maximum value at those points.

The geometric approach used in Example 6.1.3 suggests the following definitions.

Definition 6.1.2.

*(i) A constant y_c is a **critical point** of the equation $y' = f(y)$ iff holds $f(y_c) = 0$.*
*(ii) A critical point y_c is **stable** iff $f(y) > 0$ for every $y \neq y_c$ in a neighborhood of y_c.*
*(iii) A critical point y_c is **unstable** iff $f(y) < 0$ for every $y \neq y_c$ in a neighborhood of y_c.*
*(iv) A critical point y_c is **semistable** iff the point is stable on one side of the critical point and unstable on the other side.*

Remarks:

(a) Critical points are also called fixed points, stationary solutions, equilibrium solutions, critical solutions. We may use all these names in this notes. Stable points are also called attractors or sinks. Unstable points are also called repellers or sources. Semistable points are also called neutral points.
(b) That a critical point is stable means that for initial data close enough to the critical point all solutions approach the critical point as $t \to \infty$.

In Example 6.1.3 the critical points are $y_n = n\pi$. In the second graph of that example wee only marked -2π, $-\pi$, 0, π, and 2π. Filled dots represent stable critical points, and white dots represent unstable or semistable critical points. In this example all white points are unstable points.

In that second graph one can see that stable critical points have green arrows directed to them on both sides, and unstable points have arrows directed away from them on both sides. This is always the case for stable and unstable critical points. A semistable point would have one arrow pointing to the point on one side, and the other arrow pointing away from the point on the other side.

In terms of the differential equation critical points represent stationary solutions, also called t-independent solutions, or equilibrium solutions, or steady solutions. We will usually mention critical points as stationary solutions when we describe them in a yt-plane, and we reserve the name critical point when we describe them in a y-line.

On the last graph in Example 6.1.3 we have pictured the stationary solutions that are stable with a solid line, and those that are unstable with a dashed line. Semistable stationary solutions are also pictured with dashed lines. An equilibrium solutions is defined to be stable if all sufficiently small disturbances away from it damp out in time. An equilibrium solution is defined to be unstable if all sufficiently disturbances away from it grow in time.

Example 6.1.5: Find all the critical points of the first order linear system

$$y' = a\, y.$$

Study the stability of the critical points both for $a > 0$ and for $a < 0$. Sketch qualitative graphs of solutions close to the critical points.

SOLUTION: This is an equation of the form $y' = f(y)$ for $f(y) = ay$. The critical points are the constants y_c such that $0 = f(y_c) = ay_c$, so $y_c = 0$. We could now use the graphical method to study the stability of the critical point $y_c = 0$, but we do not need to do it. This equation is the particular case $b = 0$ of the equation solved in Example 6.1.2. So the solution for arbitrary initial data $y(0) = y_0$ is

$$y(t) = y_0\, e^{at}.$$

We use this expression to graph the solutions near a critical point. The result is shown below.

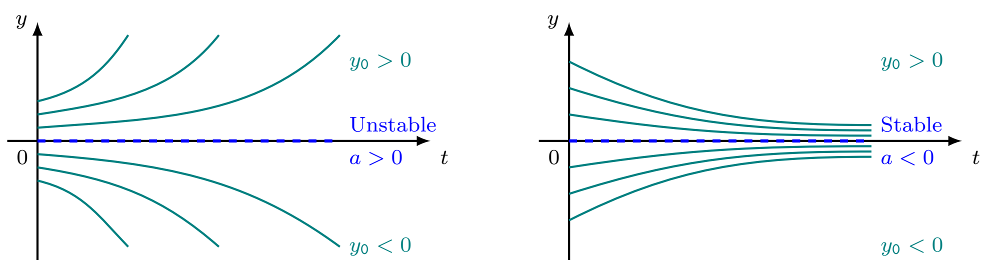


FIGURE 38. The graph of the functions $y(t) = y(0)\, e^{at}$ for $a > 0$ and $a < 0$.

We conclude that the critical point $y_c = 0$ is stable for $a < 0$ and is unstable for $a > 0$. ◁

Remark: The stability of the critical point $y_c = 0$ of the linear system $y' = ay$ will be important when we study the linearization of a nonlinear autonomous system. For that reason we highlighted these stability results in Example 6.1.5.

6.1.3. **Population Growth Models.** The simplest model for the population growth of an organism is $N' = rN$ where $N(t)$ is the population at time t and $r > 0$ is the growth rate. This model predicts exponential population growth $N(t) = N_0\, e^{rt}$, where $N_0 = N(0)$. We studied this model in § 1.5. Among other things, this model assumes that the organisms have unlimited food supply. This assumption implies that the per capita growth $N'/N = r$ is constant.

A more realistic model assumes that the per capita growth decreases linearly with N, starting with a positive value, r, and going down to zero for a critical population $N = K > 0$. So when we consider the per capita growth N'/N as a function of N, it must be given by the formula $N'/N = -(r/K)N + r$. This equation, when thought as a differential equation for N is called the logistic equation model for population growth.

Definition 6.1.3. *The **logistic equation** describes the organisms population function N in time as the solution of the autonomous differential equation*

$$N' = rN\left(1 - \frac{N}{K}\right),$$

where the initial growth rate constant r and the carrying capacity constant K are positive.

We now use the graphical method to carry out a stability analysis of the logistic population growth model. Later on we find the explicit solution of the differential equation. We can then compare the two approaches to study the solutions of the model.

EXAMPLE 6.1.6: Sketch a qualitative graph of solutions for different initial data conditions $y(0) = y_0$ to the logistic equation below, where r and K are given positive constants,

$$y' = ry\left(1 - \frac{y}{K}\right).$$

Solution:
The logistic differential equation for population growth can be written $y' = f(y)$, where function f is the polynomial

$$f(y) = ry\left(1 - \frac{y}{K}\right).$$

The first step in the graphical approach is to graph the function f. The result is in Fig. 39.
The second step is to identify all critical points of the equation. The critical points are the zeros of the function f. In this case, $f(y) = 0$ implies

$$y_0 = 0, \quad y_1 = K.$$

The third step is to find out whether the critical points are stable or unstable. Where function f is positive, a solution will be increasing, and where function f is negative a solution will be decreasing. These regions are bounded by the critical points. Now, in an interval where $f > 0$ write a right arrow, and in the intervals where $f < 0$ write a left arrow, as shown in Fig. 40.
The fourth step is to find the regions where the curvature of a solution is concave up or concave down. That information is given by y''. But the differential equation relates y'' to $f(y)$ and $f'(y)$. We have shown in Example 6.1.4 that the chain rule and the differential equation imply,

$$y'' = f'(y)\, f(y)$$

So the regions where $f(y)\, f'(y) > 0$ a solution is concave up (CU), and the regions where $f(y)\, f'(y) < 0$ a solution is concave down (CD). The result is in Fig. 41.

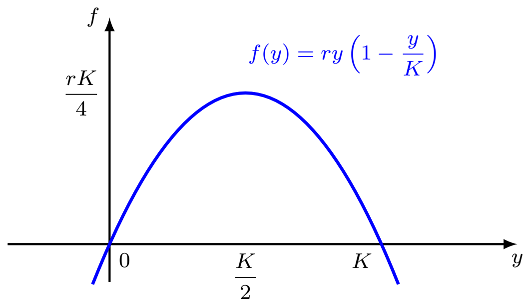


Figure 39. The graph of f.

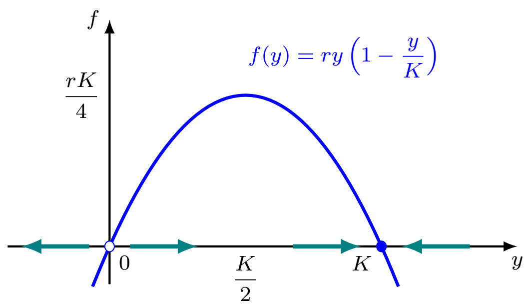


Figure 40. Critical points added.

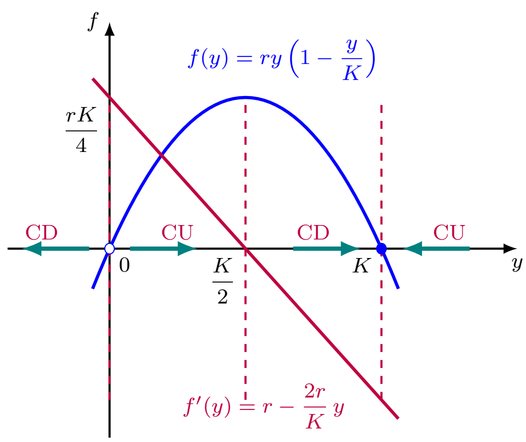


Figure 41. Concavity information added.

This is all the information we need to sketch a qualitative graph of solutions to the differential equation. So, the last step is to put all this information on a yt-plane. The horizontal axis above is now the vertical axis, and we now plot solutions y of the differential equation. The result is given in Fig. 42.

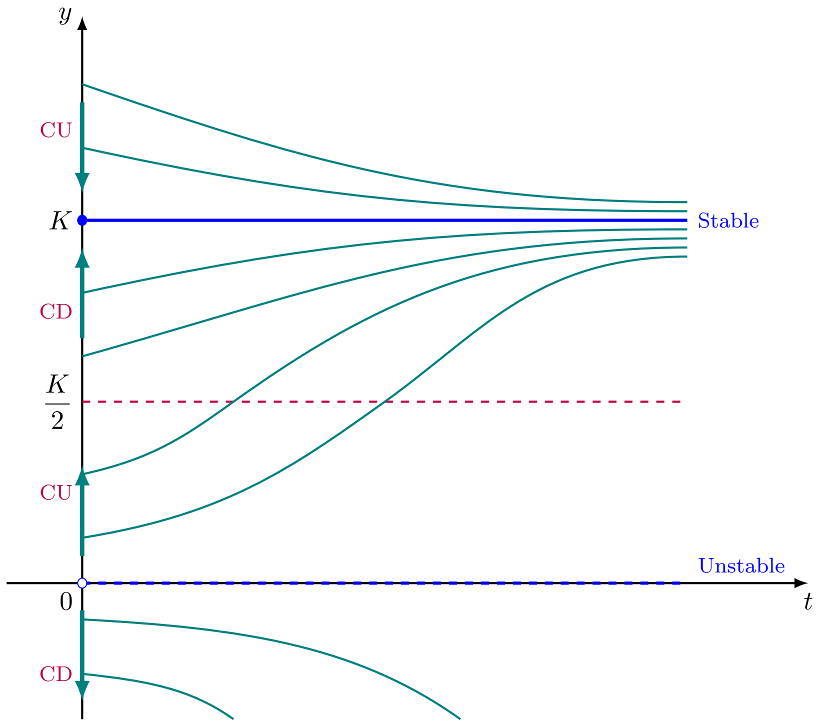


FIGURE 42. Qualitative graphs of solutions y for different initial conditions.

The picture above contains the graph of several solutions y for different choices of initial data $y(0)$. Stationary solutions are in blue, t-dependent solutions in green. The stationary solution $y_0 = 0$ is unstable and pictured with a dashed blue line. The stationary solution $y_1 = K$ is stable and pictured with a solid blue line. ◁

In Examples 6.1.4 and 6.1.6 we have used that the second derivative of the solution function is related to f and f'. This is a result that we remark here in its own statement.

Theorem 6.1.4. *If y is a solution of the autonomous system $y' = f(y)$, then*

$$y'' = f'(y)\, f(y).$$

Remark: This result has been used to find out the curvature of the solution y of an autonomous system $y' = f(y)$. The graph of y has positive curvature iff $f'(y)\, f(y) > 0$ and negative curvature iff $f'(y)\, f(y) < 0$.

Proof:

$$y'' = \frac{d}{dt}\Big(\frac{dy}{dt}\Big) = \frac{d}{dt} f(y(t)) = \frac{df}{dy}\,\frac{dy}{dt} \quad \Rightarrow \quad y'' = f'(y)\, f(y).$$

□

Remark: The logistic equation is, of course, a separable equation, so it can be solved using the method from § 1.3. We solve it below, so you can compare the qualitative graphs from Example 6.1.6 with the exact solution below.

EXAMPLE 6.1.7: Find the exact expression for the solution to the logistic equation for population growth

$$y' = ry\left(1 - \frac{y}{K}\right), \qquad y(0) = y_0, \qquad 0 < y_0 < K.$$

SOLUTION: This is a separable equation,

$$\frac{K}{r}\int \frac{y'\,dt}{(K-y)y} = t + c_0.$$

The usual substitution $u = y(t)$, so $du = y'\,dt$, implies

$$\frac{K}{r}\int \frac{du}{(K-u)u} = t + c_0.$$

We use a partial fraction decomposition on the left-hand side,

$$\frac{K}{r}\int \frac{1}{K}\Big[\frac{1}{(K-u)} + \frac{1}{u}\Big]\,du = t + c_0.$$

So each term can be integrated,

$$\big[-\ln(|K-y|) + \ln(|y|)\big] = rt + rc_0.$$

We reorder the terms on the right-hand side,

$$\ln\Big(\frac{|y|}{|K-y|}\Big) = rt + rc_0 \quad \Rightarrow \quad \Big|\frac{y}{K-y}\Big| = c\,e^{rt}, \quad c = e^{rc_0}.$$

The analysis done in Example 6.1.4 says that for initial data $0 < y_0 < K$ we can discard the absolute values in the expression above for the solution. Now the initial condition fixes the value of the constant c,

$$\frac{y_0}{K-y_0} = c.$$

Then, reordering terms we get the expression

$$y(t) = \frac{Ky_0}{y_0 + (K-y_0)\,e^{-rt}}.$$

◁

Remark: The expression above provides all solutions to the logistic equation with initial data on the interval $(0, K)$. But a stability analysis of the equation critical points is quite involved if we use that expression for the solutions. It is in this case that the geometrical analysis in Example 6.1.6 is quite useful.

6.1.4. **Linear Stability Analysis.** The geometrical analysis described above is useful to get a quick qualitative picture of solutions to an autonomous differential system. But it is always nice to complement geometric methods with analytic methods. For example, one would like an analytic way to determine the stability of a critical point. One would also like a quantitative measure of a solution decay rate to a stationary solution. A linear stability analysis can provide this type of information.

One can get information about a solution of a nonlinear equation near a critical point by studying an appropriate linear equation. More precisely, the solutions to a *nonlinear* differential equation that are close to a stationary solution can be approximated by the solutions of an appropriate *linear* differential equation. This linear equation is called the linearization of the nonlinear equation computed at the stationary solution.

Definition 6.1.5. *The* ***linearization*** *of the autonomous system $y' = f(y)$ at the critical point y_c is the linear differential system for the unknown function ξ given by*

$$\xi' = f'(y_c)\,\xi.$$

Remark: The prime notation above means, $\xi' = d\xi/dt$, and $f' = df/dy$.

EXAMPLE 6.1.8: Find the linearization of the equation $y' = \sin(y)$ at the critical point $y_n = n\pi$. Write the particular cases for $n = 0, 1$ and solve the linear equations for arbitrary initial data.

SOLUTION: If we write the nonlinear system as $y' = f(y)$, then $f(y) = \sin(y)$. We then compute its y derivative, $f'(y) = \cos(y)$. We evaluate this expression at the critical points, $f'(y_n) = \cos(n\pi) = (-1)^n$. The linearization at y_n of the nonlinear equation above is the linear equation for the unknown function ξ_n given by

$$\xi_n' = (-1)^n\,\xi_n.$$

The particular cases $n = 0$ and $n = 1$ are given by

$$\xi_0' = \xi_0, \qquad \xi_1' = -\xi_1.$$

It is simple to find solutions to first order linear homogeneous equations with constant coefficients. The result, for each equation above, is

$$\xi_0(t) = \xi_0(0)\,e^{t}, \qquad \xi_1(t) = \xi_1(0)\,e^{-t},$$

From this last expression we can see that for $n = 0$ the critical solution $\xi_0 = 0$ is unstable, while for $n = 1$ the critical solution $\xi_1 = 0$ is stable. The stability of the trivial solution $\xi_0 = x_1 = 0$ of the linearized systems coincides with the stability of the critical points $y_0 = 0$, $y_1 = \pi$ for the nonlinear equation. ◁

In the example above we have used a result that we highlight in the following statement.

Theorem 6.1.6. *The trivial solution $\xi = 0$ of the constant coefficients equation*

$$\xi' = a\,\xi$$

is stable iff $a < 0$, and it is unstable iff $a > 0$.

Proof of Theorem 6.1.6: The stability analysis follows from the explicit solutions to the differential equation, $\xi(t) = \xi(0)\,e^{at}$. For $a > 0$ the solutions diverge to $\pm\infty$ as $t \to \infty$, and for $a < 0$ the solutions approach to zero as $t \to \infty$. □

EXAMPLE 6.1.9: Find the linearization of the logistic equation $y' = ry\left(1 - \dfrac{y}{K}\right)$ at the critical points $y_0 = 0$ and $y_1 = K$. Solve the linear equations for arbitrary initial data.

SOLUTION: If we write the nonlinear system as $y' = f(y)$, then $f(y) = ry - \dfrac{r}{K}\,y^2$. Then, $f'(y) = r - \dfrac{2r}{K}\,y$. For the critical point $y_0 = 0$ we get the linearized system

$$\xi_0'(t) = r\,\xi_0 \quad \Rightarrow \quad \xi_0(t) = \xi_0(0)\,e^{rt}.$$

For the critical point $y_1 = K$ we get the linearized system

$$\xi_1'(t) = -r\,\xi_1 \quad \Rightarrow \quad \xi_1(t) = \xi_1(0)\,e^{-rt}.$$

From this last expression we can see that for $y_0 = 0$ the critical solution $\xi_0 = 0$ is unstable, while for $y_1 = K$ the critical solution $\xi_1 = 0$ is stable. The stability of the trivial solution $\xi_0 = \xi_1 = 0$ of the linearized system coincides with the stability of the critical points $y_0 = 0$, $y_1 = K$ for the nonlinear equation. ◁

Remark: In the Examples 6.1.8 and 6.1.9 we have seen that the stability of a critical point y_c to a nonlinear differential equation $y' = f(y)$ is the same as the stability of the trivial solution $\xi = 0$ of the linearized equation $\xi' = f'(y_c)\,\xi$. This is a general result, which we state below.

Theorem 6.1.7. *Let y_c be a critical point of the autonomous system $y' = f(y)$.*
(a) The critical point y_c is stable iff $f'(y_c) < 0$.
(b) The critical point y_c is unstable iff $f'(y_c) > 0$.
Furthermore, If the initial data $y(0) \simeq y_c$, is close enough to the critial point y_c, then the solution with that initial data of the equation $y' = f(y)$ are close enough to y_c in the sense

$$y(t) \simeq y_c + \xi(t),$$

where ξ is the solution to the linearized equation at the critical point y_c,

$$\xi' = f'(y_c)\,\xi, \qquad \xi(0) = y(0) - y_c.$$

Remark: The proof of this result can be found in § 43 in Simmons' textbook [10].

Remark: The first part of Theorem 6.1.7 highlights the importance of the sign fo the coefficient $f'(y_c)$, which determines the stability of the critical point y_c. The furthermore part of the Theorem highlights how stable is a critical point. The value $|f'(y_c)|$ plays a role of an exponential growth or a exponential decay rate. Its reciprocal, $1/|f'(y_c)|$ is a *characteristic scale*. It determines the value of t required for the solution y to vary significantly in a neighborhood of the critical point y_c.

Notes

This section follows a few parts of Chapter 2 in Steven Strogatz's book on Nonlinear Dynamics and Chaos, [12], and also § 2.5 in Boyce DiPrima classic textbook [3].

6.1.5. Exercises.

6.1.1.- .

6.1.2.- .

6.2. Flows on the Plane

Coming up.

6.3. Linear Stability

Coming up.

Chapter 7. Boundary Value Problems

7.1. Eigenvalue-Eigenfunction Problems

In this Section we consider second order, linear, ordinary differential equations. In the first half of the Section we study boundary value problems for these equations and in the second half we focus on a particular type of boundary value problems, called the eigenvalue-eigenfunction problem for these equations.

7.1.1. **Comparison: IVP and BVP.**

Given real constants a_1 and a_0, consider the second order, linear, homogeneous, constant coefficients, ordinary differential equation

$$y'' + a_1\, y' + a_0\, y = 0. \tag{7.1.1}$$

We now review the initial boundary value problem for the equation above, which was discussed in Sect. **??**, where we showed in Theorem **??** that this initial value problem always has a unique solution.

Definition 7.1.1 (IVP). *Given the constants t_0, y_0 and y_1, find a solution y of Eq. (7.1.1) satisfying the initial conditions*

$$y(t_0) = y_0, \qquad y'(t_0) = y_1. \tag{7.1.2}$$

There are other problems associated to the differential equation above. The following one is called a boundary value problem.

Definition 7.1.2 (BVP). *Given the constants $t_0 \neq t_1$, y_0 and y_1, find a solution y of Eq. (7.1.1) satisfying the boundary conditions*

$$y(t_0) = y_0, \qquad y(t_1) = y_1. \tag{7.1.3}$$

One could say that the origins of the names "initial value problem" and "boundary value problem" originates in physics. Newton's second law of motion for a point particle was the differential equation to solve in an initial value problem; the unknown function y was interpreted as the position of the point particle; the independent variable t was interpreted as time; and the additional conditions in Eq. (7.1.2) were interpreted as specifying the position and velocity of the particle at an initial time. In a boundary value problem, the differential equation was any equation describing a physical property of the system under study, for example the temperature of a solid bar; the unknown function y represented any physical property of the system, for example the temperature; the independent variable t represented position in space, and it is usually denoted by x; and the additional conditions given in Eq. (7.1.3) represent conditions on the physical quantity y at two different positions in space given by t_0 and t_1, which are usually the boundaries of the system under study, for example the temperature at the boundaries of the bar. This originates the name "boundary value problem".

We mentioned above that the initial value problem for Eq. (7.1.1) always has a unique solution for every constants y_0 and y_1, result presented in Theorem **??**. The case of the boundary value problem for Eq. (7.1.1) is more complicated. A boundary value problem may have a unique solution, or may have infinitely many solutions, or may have no solution, depending on the boundary conditions. This result is stated in a precise way below.

Theorem 7.1.3 (BVP). *Fix real constants a_1, a_0, and let $r_\pm$ be the roots of the characteristic polynomial $p(r) = r^2 + a_1 r + a_0$.*

(i) If the roots $r_\pm \in \mathbb{R}$, then the boundary value problem given by Eqs. (7.1.1) and (7.1.3) has a unique solution for all y_0, $y_1 \in \mathbb{R}$.

(ii) If the roots $r_\pm$ form a complex conjugate pair, that is, $r_\pm = \alpha \pm \beta i$, with α, $\beta \in \mathbb{R}$, then the solution of the boundary value problem given by Eqs. (7.1.1) and (7.1.3) belongs to only one of the following three possibilities:
(a) There exists a unique solution;
(b) There exists infinitely many solutions;
(c) There exists no solution.

Before presenting the proof of Theorem 7.1.3 concerning boundary value problem, let us review part of the proof of Theorem **??** concerning initial value problems using matrix notation to highlight the crucial part of the calculations. For the simplicity of this review, we only consider the case $r_+ \neq r_-$. In this case the general solution of Eq. (7.1.1) can be expressed as follows,

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}.$$

The initial conditions in Eq. (7.1.2) determine the values of the constants c_1 and c_2 as follows:

$$\left.\begin{aligned} y_0 = y(t_0) &= c_1\, e^{r_- t_0} + c_2\, e^{r_+ t_0} \\ y_1 = y'(t_0) &= c_1 r_-\, e^{r_- t_0} + c_2 r_+\, e^{r_+ t_0} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the determinant of the coefficient matrix Z is non-zero, where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ r_-\, e^{r_- t_0} & r_+\, e^{r_+ t_0} \end{bmatrix}.$$

A simple calculation shows

$$\det(Z) = \big(r_+ - r_-\big)\, e^{(r_+ + r_-)\, t_0} \neq 0 \quad \Leftrightarrow \quad r_+ \neq r_-.$$

Since $r_+ \neq r_-$, the matrix Z is invertible and so the initial value problem above a unique solution for every choice of the constants y_0 and y_1. The proof of Theorem 7.1.3 for the boundary value problem follows the same steps we described above: First find the general solution to the differential equation, second find whether the matrix Z above is invertible or not.

Proof of Theorem 7.1.3:

Part (i): Assume that $r_\pm$ are real numbers. We have two cases, $r_+ \neq r_-$ and $r_+ = r_-$. In the former case the general solution to Eq. (7.1.1) is given by

$$y(t) = c_1\, e^{r_- t} + c_2\, e^{r_+ t}, \qquad c_1, c_2 \in \mathbb{R}. \tag{7.1.4}$$

The boundary conditions in Eq. (7.1.3) determine the values of the constants c_1, c_2, since

$$\left.\begin{aligned} y_0 = y(t_0) &= c_1 e^{r_- t_0} + c_2 e^{r_+ t_0} \\ y_1 = y(t_1) &= c_1 e^{r_- t_1} + c_2 e^{r_+ t_1} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ e^{r_- t_1} & e^{r_+ t_1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}. \tag{7.1.5}$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the determinant of the coefficient matrix Z is non-zero, where

$$Z = \begin{bmatrix} e^{r_- t_0} & e^{r_+ t_0} \\ e^{r_- t_1} & e^{r_+ t_1} \end{bmatrix}. \tag{7.1.6}$$

A straightforward calculation shows

$$\det(Z) = e^{r_+ t_1} e^{r_- t_0} - e^{r_+ t_0} e^{r_- t_1} = e^{r_+ t_0} e^{r_- t_0} \big[e^{r_+ (t_1 - t_0)} - e^{r_- (t_1 - t_0)}\big]. \tag{7.1.7}$$

So it is simple to verify that

$$\det(Z) \neq 0 \quad \Leftrightarrow \quad e^{r_+ (t_1 - t_0)} \neq e^{r_- (t_1 - t_0)} \quad \Leftrightarrow \quad r_+ \neq r_-. \tag{7.1.8}$$

Therefore, in the case $r_+ \neq r_-$ the matrix Z is invertible and so the boundary value problem above has a unique solution for every choice of the constants y_0 and y_1. In the case that

$r_+ = r_- = r_0$, then we have to start over, since the general solution of Eq. (7.1.1) is not given by Eq. (7.1.4) but by the following expression

$$y(t) = (c_1 + c_2\, t)\, e^{r_0 t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Again, the boundary conditions in Eq. (7.1.3) determine the values of the constants c_1 and c_2 as follows:

$$\left.\begin{aligned} y_0 = y(t_0) = c_1 e^{r_0 t_0} + c_2 t_0 e^{r_0 t_0} \\ y_1 = y(t_1) = c_1 e^{r_0 t_1} + c_2 t_1 e^{r_0 t_1} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} e^{r_0 t_0} & t_0 e^{r_0 t_0} \\ e^{r_0 t_1} & t_1 e^{r_0 t_1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}.$$

The linear system above has a unique solution c_1 and c_2 for every constants y_0 and y_1 iff the determinant of the coefficient matrix Z is non-zero, where

$$Z = \begin{bmatrix} e^{r_0 t_0} & t_0 e^{r_0 t_0} \\ e^{r_0 t_1} & t_1 e^{r_0 t_1} \end{bmatrix}$$

A simple calculation shows

$$\det(Z) = t_1 e^{r_0 (t_1 + t_0)} - t_0 e^{r_0 (t_1 + t_0)} = (t_1 - t_0) e^{r_0 (t_1 + t_0)} \neq 0 \quad \Leftrightarrow \quad t_1 \neq t_0.$$

Therefore, in the case $r_+ = r_- = r_0$ the matrix Z is again invertible and so the boundary value problem above has a unique solution for every choice of the constants y_0 and y_1. This establishes part (i) of the Theorem.

Part (ii): Assume that the roots of the characteristic polynomial have the form $r_\pm = \alpha \pm \beta i$ with $\beta \neq 0$. In this case the general solution to Eq. (7.1.1) is still given by Eq. (7.1.4), and the boundary condition of the problem are still given by Eq. (7.1.5). The determinant of matrix Z introduced in Eq. (7.1.6) is still given by Eq. (7.1.7). However, the claim given in Eq. (7.1.8) is true only in the case that $r_\pm$ are real numbers, and it does not hold in the case that $r_\pm$ are complex numbers. The reason is the following calculation: Let us start with Eq. (7.1.7) and then introduce that $r_\pm = \alpha \pm \beta i$;

$$\begin{aligned} \det(Z) &= e^{(r_+ + r_-)\, t_0} \big[e^{r_+\, (t_1 - t_0)} - e^{r_-\, (t_1 - t_0)} \big] \\ &= e^{2\alpha t_0}\, e^{\alpha (t_1 - t_0)}\, \big[e^{i\beta (t_1 - t_0)} - e^{-i\beta (t_1 - t_0)} \big] \\ &= 2i\, e^{\alpha (t_1 + t_0)}\, \sin\big[\beta (t_1 - t_0)\big]. \end{aligned}$$

We then conclude that

$$\det(Z) = 0 \quad \Leftrightarrow \quad \sin\big[\beta(t_1 - t_0)\big] = 0 \quad \Leftrightarrow \quad \beta = \frac{n\pi}{(t_1 - t_0)}, \tag{7.1.9}$$

where $n = 1, 2, \cdots$ and we are using that $t_1 \neq t_0$. This last equation in (7.1.9) is the key to obtain all three cases in part (ii) of this Theorem, as can be seen from the following argument:

Part (iia): If the coefficients a_1, a_0 in Eq. (7.1.1) and the numbers t_1, t_0 in the boundary conditions in (7.1.3) are such that Eq. (7.1.9) does not hold, that is,

$$\beta \neq \frac{n\pi}{(t_1 - t_0)},$$

then $\det(Z) \neq 0$ and so the boundary value problem for Eq. (7.1.1) has a unique solution for all constants y_0 and y_1. This establishes part (iia).

Part (iib) and Part (iic): If the coefficients a_1, a_0 in Eq. (7.1.1) and the numbers t_1, t_0 in the boundary conditions in (7.1.3) are such that Eq. (7.1.9) holds, then $\det(Z) = 0$, and so the system of linear equation given in (7.1.5) may or may not have solutions, depending on the values of the constants y_0 and y_1. In the case that there exists a solution, then there are infinitely many solutions, since $\det(Z) = 0$. This establishes part (iib). The remaining case, when y_0 and y_1 are such that Eq. (7.1.5) has no solution is the case in part (iic). This establishes the Theorem. □

Our first example is a boundary value problem with a unique solution. This corresponds to case (iia) in Theorem 7.1.3. The matrix Z defined in the proof of that Theorem is invertible and the boundary value problem has a unique solution for every y_0 and y_1.

EXAMPLE 7.1.1: Find the solution $y(x)$ to the boundary value problem

$$y'' + 4y = 0, \qquad y(0) = 1, \quad y(\pi/4) = -1.$$

SOLUTION: We first find the general solution to the differential equation above. We know that we have to look for solutions of the form $y(x) = e^{rx}$, with the constant r being solutions of the characteristic equation

$$r^2 + 4 = 0 \quad \Leftrightarrow \quad r_\pm = \pm 2i.$$

We know that in this case we can express the general solution to the differential equation above as follows,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

The boundary conditions imply the following system of linear equation for c_1 and c_2,

$$\left.\begin{aligned} 1 = y(0) = c_1 \\ -1 = y(\pi/4) = c_2 \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The linear system above has the unique solution $c_1 = 1$ and $c_2 = -1$. Hence, the boundary value problem above has the unique solution

$$y(x) = \cos(2x) - \sin(2x).$$

◁

The following example is a small variation of the previous one, we change the value of the constant t_1, which is the place where we impose the second boundary condition, from $\pi/4$ to $\pi/2$. This is enough to have a boundary value problem with infinitely many solutions, corresponding to case (iib) in Theorem 7.1.3. The matrix Z in the proof of this Theorem 7.1.3 is not invertible in this case, and the values of the constants t_0, t_1, y_0, y_1, a_1 and a_0 are such that there are infinitely many solutions.

EXAMPLE 7.1.2: Find the solution $y(x)$ to the boundary value problem

$$y'' + 4y = 0, \qquad y(0) = 1, \quad y(\pi/2) = -1.$$

SOLUTION: The general solution is the same as in Example 7.1.1 above, that is,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

The boundary conditions imply the following system of linear equation for c_1 and c_2,

$$\left.\begin{aligned} 1 = y(0) = c_1 \\ -1 = y(\pi/2) = -c_1 \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The linear system above has infinitely many solution, as can be seen from the following:

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ -1 & 0 & -1 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{cases} c_1 = 1, \\ c_2 \quad \text{free.} \end{cases}$$

Hence, the boundary value problem above has infinitely many solutions given by

$$y(x) = \cos(2x) + c_2 \sin(2x), \qquad c_2 \in \mathbb{R}.$$

◁

The following example again is a small variation of the previous one, this time we change the value of the constant y_1 from -1 to 1. This is enough to have a boundary value problem with no solutions, corresponding to case (iic) in Theorem 7.1.3. The matrix Z in the proof of this Theorem 7.1.3 is still not invertible, and the values of the constants t_0, t_1, y_0, y_1, a_1 and a_0 are such that there is not solution.

Example 7.1.3: Find the solution y to the boundary value problem

$$y''(x) + 4y(x) = 0, \qquad y(0) = 1, \quad y(\pi/2) = 1.$$

Solution: The general solution is the same as in Examples 7.1.1 and 7.1.2 above, that is,

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

The boundary conditions imply the following system of linear equation for c_1 and c_2,

$$\begin{aligned} 1 &= y(0) = c_1 \\ 1 &= y(\pi/2) = -c_1 \end{aligned}$$

From the equations above we see that there is no solution for c_1, hence there is no solution for the boundary value problem above.

Remark: We now use matrix notation, in order to follow the same steps we did in the proof of Theorem 7.1.3:

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The linear system above has infinitely many solutions, as can be seen from the following Gauss elimination operations

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

Hence, there are no solutions to the linear system above. ◁

7.1.2. Eigenvalue-eigenfunction problems.

A particular type of boundary value problems are called eigenvalue-eigenfunction problems. The main example we study in this Section is the following: Find all the numbers λ and the non-zero functions with values $y(x)$ solutions of the homogeneous boundary value problem

$$y'' = \lambda y, \qquad y(0) = 0, \quad y(\ell) = 0, \quad \ell > 0. \tag{7.1.10}$$

This problem is analogous to the eigenvalue-eigenvector problem studied in Sect. 8.3, that is, given an $n \times n$ matrix A find all numbers λ and non-zero vectors $\boldsymbol{v}$ solution of the algebraic linear system $A\boldsymbol{v} = \lambda \boldsymbol{v}$. The role of matrix A is played by d^2/dx^2, the role of the vector space $\mathbb{R}^n$ is played by the vector space of all infinitely differentiable functions f with domain $[0, \ell] \subset \mathbb{R}$ satisfying $f(0) = f(\ell) = 0$. We mentioned in Sect. 8.3 that given any $n \times n$ matrix A there exist at most n eigenvalues and eigenvectors. In the case of the boundary value problem in Eq. (7.1.10) there exist infinitely many solutions λ and $y(x)$, as can be seen in the following result.

Theorem 7.1.4 (Eigenvalues-eigenfunctions). *The homogeneous boundary value problem in Eq. (7.1.10) has the infinitely many solutions, labeled by a subindex $n \in \mathbb{N}$,*

$$\lambda_n = -\frac{n^2 \pi^2}{\ell^2}, \qquad y_n(x) = \sin\Big(\frac{n\pi x}{\ell}\Big).$$

Proof of Theorem 7.1.4: We first look for solutions having eigenvalue $\lambda = 0$. In such a case the general solution to the differential equation in (7.1.10) is given by

$$y(x) = c_1 + c_2 x, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{array}{l} 0 = y(0) = c_1, \\ 0 = y(\ell) = c_1 + c_2\ell \end{array}\right\} \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Since the only solution in this case is $y = 0$, there are no non-zero solutions.

We now look for solutions having eigenvalue $\lambda > 0$. In this case we redefine the eigenvalue as $\lambda = \mu^2$, with $\mu > 0$. The general solution to the differential equation in (7.1.10) is given by

$$y(x) = c_1 e^{-\mu x} + c_2 e^{\mu x},$$

where we used that the roots of the characteristic polynomial $r^2 - \mu^2 = 0$ are given by $r_\pm = \pm\mu$. The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{array}{l} 0 = y(0) = c_1 + c_2, \\ 0 = y(\ell) = c_1 e^{-\mu\ell} + c_2 e^{\mu\ell} \end{array}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ e^{-\mu\ell} & e^{\mu\ell} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Denoting by

$$Z = \begin{bmatrix} 1 & 1 \\ e^{-\mu\ell} & e^{\mu\ell} \end{bmatrix}$$

we see that

$$\det(Z) = e^{\mu\ell} - e^{-\mu\ell} \neq 0 \quad \Leftrightarrow \quad \mu \neq 0.$$

Hence the matrix Z is invertible, and then we conclude that the linear system above for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$, and so $y = 0$. Therefore there are no non-zero solutions y in the case that $\lambda > 0$.

We now study the last case, when the eigenvalue $\lambda < 0$. In this case we redefine the eigenvalue as $\lambda = -\mu^2$, with $\mu > 0$, and the general solution to the differential equation in (7.1.10) is given by

$$y(x) = \tilde{c}_1 e^{-i\mu x} + \tilde{c}_2 e^{i\mu x},$$

where we used that the roots of the characteristic polynomial $r^2 + \mu^2 = 0$ are given by $r_\pm = \pm i\mu$. In a case like this one, when the roots of the characteristic polynomial are complex, it is convenient to express the general solution above as a linear combination of real-valued functions,

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{array}{l} 0 = y(0) = c_1, \\ 0 = y(\ell) = c_1 \cos(\mu\ell) + c_2 \sin(\mu\ell) \end{array}\right\} \quad \Rightarrow \quad c_2 \sin(\mu\ell) = 0.$$

Since we are interested in non-zero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\sin(\mu\ell) = 0 \quad \Leftrightarrow \quad \mu_n \ell = n\pi, \qquad n \in \mathbb{N}.$$

We therefore conclude that the eigenvalues and eigenfunctions are given by

$$\lambda_n = -\frac{n^2\pi^2}{\ell^2}, \qquad y_n(x) = c_n \sin\Bigl(\frac{n\pi x}{\ell}\Bigr).$$

Choosing the free constants $c_n = 1$ we establish the Theorem. $\square$

Example 7.1.4: Find the numbers λ and the non-zero functions with values $y(x)$ solutions of the following homogeneous boundary value problem

$$y'' = \lambda y, \qquad y(0) = 0, \qquad y'(\pi) = 0.$$

Solution: This is also an eigenvalue-eigenfunction problem, the only difference with the case studied in Theorem 7.1.4 is that the second boundary condition here involves the derivative of the unknown function y. The solution is obtained following exactly the same steps performed in the proof of Theorem 7.1.4.

We first look for solutions having eigenvalue $\lambda = 0$. In such a case the general solution to the differential equation is given by

$$y(x) = c_1 + c_2 x, \qquad c_1, c_2 \in \mathbb{R}.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$0 = y(0) = c_1, \qquad 0 = y'(\pi) = c_2.$$

Since the only solution in this case is $y = 0$, there are no non-zero solutions with $\lambda = 0$.

We now look for solutions having eigenvalue $\lambda > 0$. In this case we redefine the eigenvalue as $\lambda = \mu^2$, with $\mu > 0$. The general solution to the differential equation is given by

$$y(x) = c_1 e^{-\mu x} + c_2 e^{\mu x},$$

where we used that the roots of the characteristic polynomial $r^2 - \mu^2 = 0$ are given by $r_\pm = \pm\mu$. The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(0) = c_1 + c_2, \\ 0 &= y'(\pi) = -\mu c_1 e^{-\mu\pi} + \mu c_2 e^{\mu\pi} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ -\mu e^{-\mu\pi} & \mu e^{\mu\pi} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Denoting by

$$Z = \begin{bmatrix} 1 & 1 \\ -\mu e^{-\mu\pi} & \mu e^{\mu\pi} \end{bmatrix}$$

we see that

$$\det(Z) = \mu\big(e^{\mu\pi} + e^{-\mu\pi}\big) \neq 0.$$

Hence the matrix Z is invertible, and then we conclude that the linear system above for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$, and so $y = 0$. Therefore there are no non-zero solutions y in the case that $\lambda > 0$.

We now study the last case, when the eigenvalue $\lambda < 0$. In this case we redefine the eigenvalue as $\lambda = -\mu^2$, with $\mu > 0$, and the general solution to the differential equation in (7.1.10) is given by

$$y(x) = \tilde{c}_1 e^{-i\mu x} + \tilde{c}_2 e^{i\mu x},$$

where we used that the roots of the characteristic polynomial $r^2 + \mu^2 = 0$ are given by $r_\pm = \pm i\mu$. As we did in the proof of Theorem 7.1.4, it is convenient to express the general solution above as a linear combination of real-valued functions,

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(0) = c_1, \\ 0 &= y'(\pi) = -\mu c_1 \sin(\mu\pi) + \mu c_2 \cos(\mu\pi) \end{aligned}\right\} \quad \Rightarrow \quad c_2 \cos(\mu\pi) = 0.$$

Since we are interested in non-zero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\cos(\mu\pi) = 0 \quad \Leftrightarrow \quad \mu_n \pi = (2n+1)\frac{\pi}{2}, \qquad n \in \mathbb{N}.$$

We therefore conclude that the eigenvalues and eigenfunctions are given by

$$\lambda_n = -\frac{(2n+1)^2}{4}, \qquad y_n(x) = c_n \sin\Big[\frac{(2n+1)x}{2}\Big], \qquad n \in \mathbb{N}.$$

$\lhd$

EXAMPLE 7.1.5: Find the numbers λ and the non-zero functions with values $y(x)$ solutions of the homogeneous boundary value problem

$$x^2\, y'' - x\, y' = \lambda\, y, \qquad y(1) = 0, \quad y(\ell) = 0, \quad \ell > 1.$$

SOLUTION: This is also an eigenvalue-eigenfunction problem, the only difference with the case studied in Theorem 7.1.4 is that the differential equation is now the Euler equation, studied in Sect. 3.2, instead of a constant coefficient equation. Nevertheless, the solution is obtained following exactly the same steps performed in the proof of Theorem 7.1.4.

Writing the differential equation above in the standard form of an Euler equation,

$$x^2\, y'' - x\, y' - \lambda y = 0,$$

we know that the general solution to the Euler equation is given by

$$y(x) = \big[c_1 + c_2 \ln(x)\big]\, x^{r_0}$$

in the case that the constants $r_+ = r_- = r_0$, where $r_\pm$ are the solutions of the Euler characteristic equation

$$r(r-1) - r - \lambda = 0 \quad \Rightarrow \quad r_\pm = 1 \pm \sqrt{1+\lambda}.$$

In the case that $r_+ \neq r_-$, then the general solution to the Euler equation has the form

$$y(x) = c_1 x^{r_-} + c_2 x^{r_+}.$$

Let us start with the first case, when the roots of the Euler characteristic polynomial are repeated $r_+ = r_- = r_0$. In our case this happens if $1 + \lambda = 0$. In such a case $r_0 = 1$, and the general solution to the Euler equation is

$$y(x) = \big[c_1 + c_2 \ln(x)\big]\, x.$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1, \\ 0 &= y(\ell) = \big[c_1 + c_2 \ln(\ell)\big]\,\ell \end{aligned}\right\} \quad \Rightarrow \quad c_2 \ell \ln(\ell) = 0,$$

hence $c_2 = 0$. We conclude that the linear system above for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$, and so $y = 0$. Therefore there are no non-zero solutions y in the case that $1 + \lambda = 0$.

We now look for solutions having eigenvalue λ satisfying the condition $1 + \lambda > 0$. In this case we redefine the eigenvalue as $1 + \lambda = \mu^2$, with $\mu > 0$. Then, $r_\pm = 1 \pm \mu$, and so the general solution to the differential equation is given by

$$y(x) = c_1 x^{(1-\mu)} + c_2 x^{(1+\mu)},$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1 + c_2, \\ 0 &= y(\ell) = c_1 \ell^{(1-\mu)} + c_2 \ell^{(1+\mu)} \end{aligned}\right\} \quad \Rightarrow \quad \begin{bmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Denoting by

$$Z = \begin{bmatrix} 1 & 1 \\ \ell^{(1-\mu)} & \ell^{(1+\mu)} \end{bmatrix}$$

we see that

$$\det(Z) = \ell\big(\ell^{\mu} - \ell^{-\mu}\big) \neq 0 \quad \Leftrightarrow \quad \ell \neq \pm 1.$$

Hence the matrix Z is invertible, and then we conclude that the linear system above for c_1, c_2 has a unique solution given by $c_1 = c_2 = 0$, and so $y = 0$. Therefore there are no non-zero solutions y in the case that $1 + \lambda > 0$.

We now study the second case, when the eigenvalue satisfies that $1 + \lambda < 0$. In this case we redefine the eigenvalue as $1 + \lambda = -\mu^2$, with $\mu > 0$. Then $r_\pm = 1 \pm i\mu$, and the general solution to the differential equation is given by

$$y(x) = \tilde{c}_1 x^{(1-i\mu)} + \tilde{c}_2 x^{(1+i\mu)},$$

As we did in the proof of Theorem 7.1.4, it is convenient to express the general solution above as a linear combination of real-valued functions,

$$y(x) = x\big[c_1 \cos\big(\mu \ln(x)\big) + c_2 \sin\big(\mu \ln(x)\big)\big].$$

The boundary conditions imply the following conditions on c_1 and c_2,

$$\left.\begin{aligned} 0 &= y(1) = c_1, \\ 0 &= y(\ell) = c_1 \ell \cos\big[\mu \ln(\ell)\big] + c_2 \ell \sin\big[(\mu \ln(\ell)\big] \end{aligned}\right\} \quad \Rightarrow \quad c_2 \ell \sin\big[\mu \ln(\ell)\big] = 0.$$

Since we are interested in non-zero solutions y, we look for solutions with $c_2 \neq 0$. This implies that μ cannot be arbitrary but must satisfy the equation

$$\sin\big[\mu \ln(\ell)\big] = 0 \quad \Leftrightarrow \quad \mu_n \ln(\ell) = n\pi, \qquad n \in \mathbb{N}.$$

We therefore conclude that the eigenvalues and eigenfunctions are given by

$$\lambda_n = -1 - \frac{n^2\pi^2}{\ln^2(\ell)}, \qquad y_n(x) = c_n x \sin\Big[\frac{n\pi \ln(x)}{\ln(\ell)}\Big], \qquad n \in \mathbb{N}.$$

$\lhd$

7.1.3. Exercises.

7.1.1.- .

7.1.2.- .

7.2. Overview of Fourier Series

This Section is a brief introduction to the Fourier series expansions of periodic functions. We first recall the origins of this type of series expansions. We then review basic few notions of linear algebra that are satisfied by the set of infinitely differentiable functions. One crucial concept is the orthogonality of the sine and cosine functions. We then introduce the Fourier series of periodic functions, and we end this Section with the study two particular cases: The Fourier series of odd and of even functions, which are called sine and cosine series, respectively.

7.2.1. **Origins of Fourier series.** The study of solutions to the wave equation in one space dimension by Daniel Bernoulli in the 1750s is a possible starting point to describe the origins of the Fourier series. The physical system is a vibrating elastic string with fixed ends, the unknown function with values $u(t,x)$ represents the vertical displacement of a point in the string at the time t and position x, as can be seen in the sketch given in Fig. 43. A constant $c > 0$ characterizes the material that form the string.

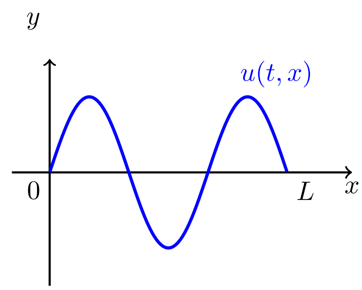


Figure 43. Vibrating string moving on the vertical direction with fixed ends.

The mathematical problem to solve is the following initial-boundary value problem: Given a function with values $f(x)$ defined in the interval $[0,\ell] \subset \mathbb{R}$ satisfying $f(0) = f(\ell) = 0$, find a function with values $u(t,x)$ solution of the wave equation

$$\begin{gathered} \partial_t^2 u(t,x) = c^2\, \partial_x^2 u(t,x), \\ u(t,0) = 0, \quad u(t,\ell) = 0, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = 0. \end{gathered}$$

The equations on the second line are called boundary conditions, since they are conditions at the boundary of the vibrating string for all times. The equations on the third line are called initial conditions, since they are equation that hold at the initial time only. The first equation says that the initial position of the string is given by the function f, while the second equation says that the initial velocity of the string vanishes. Bernoulli found that the functions

$$u_n(t,x) = \cos\Big(\frac{cn\pi t}{\ell}\Big)\, \sin\Big(\frac{n\pi x}{\ell}\Big)$$

are particular solutions to the problem above in the case that the initial position function is given by

$$f_n(x) = \sin\Big(\frac{n\pi x}{\ell}\Big).$$

He also found that the function

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, \cos\Big(\frac{cn\pi t}{\ell}\Big)\, \sin\Big(\frac{n\pi x}{\ell}\Big)$$

is also a solution to the problem above with initial condition

$$f(x) = \sum_{n=1}^{\infty} c_n\, \sin\Big(\frac{n\pi x}{\ell}\Big). \tag{7.2.1}$$

Is the set of initial functions f given in Eq. (7.2.1) big enough to include all continuous functions satisfying the compatibility conditions $f(0) = f(\ell) = 0$? Bernoulli said the answer was yes, and his argument was that with infinitely many coefficients c_n one can compute every function f satisfying the compatibility conditions. Unfortunately this argument does not prove Bernoulli's claim. A proof would be a formula to compute the coefficients c_n in terms of the function f. However, Bernoulli could not find such a formula.

A formula was obtained by Joseph Fourier in the 1800s while studying a different problem. He was looking for solutions to the following initial-boundary value problem: Given a function with values $f(x)$ defined in the interval $[0, \ell] \subset \mathbb{R}$ satisfying $f(0) = f(\ell) = 0$, and given a positive constant k, find a function with values $u(t, x)$ solution of the differential equation

$$\begin{gathered} \partial_t u(t,x) = k\,\partial_x^2 u(t,x), \\ u(t,0) = 0, \quad u(t,\ell) = 0, \\ u(0,x) = f(x). \end{gathered}$$

The values of the unknown function $u(t, x)$ are interpreted as the temperature of a solid body at the time t and position x. The temperature in this problem does not depend on the y and z coordinates. The partial differential equation on the first line above is called the heat equation, and describes the variation of the body temperature. The thermal properties of the body material are specified by the positive constant k, called the thermal diffusivity. The main difference with the wave equation above is that only first time derivatives appear in the equation. The boundary conditions on the second line say that both borders of the body are held at constant temperature. The initial condition on the third line provides the initial temperature of the body. Fourier found that the functions

$$u_n(t,x) = e^{-(\frac{n\pi}{\ell})^2 kt} \sin\Big(\frac{n\pi x}{\ell}\Big)$$

are particular solutions to the problem above in the case that the initial position function is given by

$$f_n(x) = \sin\Big(\frac{n\pi x}{\ell}\Big).$$

Fourier also found that the function

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-(\frac{n\pi}{\ell})^2 kt} \sin\Big(\frac{n\pi x}{\ell}\Big)$$

is also a solution to the problem above with initial condition

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{\ell}\Big). \tag{7.2.2}$$

Fourier was able to show that any continuous function f defined on the domain $[0, \ell] \subset \mathbb{R}$ satisfying the conditions $f(0) = f(\ell) = 0$ can be written as the series given in Eq. (7.2.2), where the coefficients c_n can be computed with the formula

$$c_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\Big(\frac{n\pi x}{\ell}\Big)\, dx.$$

This formula for the coefficients c_n, together with few other formulas that we will study later on in this Section, was an important reason to name after Fourier instead of Bernoulli the series containing those given in Eq. (7.2.2).

7.2.2. **Fourier series.** Every continuous τ-periodic function f can be expressed as an infinite linear combination of sine and cosine functions. Before we present this result in a precise form we need to introduce few definitions and to recall few concepts from linear algebra. We start defining a peridic function saying that it is invariant under certain translations.

Definition 7.2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is called **τ-periodic** iff for all $x \in \mathbb{R}$ holds*

$$f(x - \tau) = f(x), \qquad \tau > 0.$$

The number τ is called the *period* of f, and the definition says that a function τ-periodic iff it is invariant under translations by τ and so, under translations by any multiple of τ.

Example 7.2.1: The following functions are periodic, with period τ,

$$\begin{aligned} f(x) &= \sin(x), & \tau &= 2\pi,\\ f(x) &= \cos(x), & \tau &= 2\pi,\\ f(x) &= \tan(x), & \tau &= \pi,\\ f(x) &= \sin(ax), & \tau &= \frac{2\pi}{a}. \end{aligned}$$

The following function is also periodic and its graph is given in Fig. 44,

$$f(x) = e^x, \qquad x \in [0, 2), \qquad f(x - 2) = f(x). \tag{7.2.3}$$

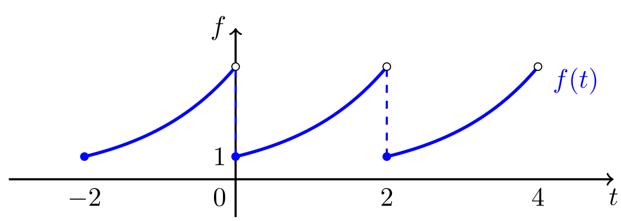


Figure 44. The graph of the function given in Eq. (7.2.3).

◁

Example 7.2.2: Show that the following functions are τ-periodic for all $n \in \mathbb{N}$,

$$f_n(x) = \cos\Big(\frac{2\pi n x}{\tau}\Big), \qquad g_n(x) = \sin\Big(\frac{2\pi n x}{\tau}\Big).$$

Solution: The following calculation shows that f_n is τ-periodic,

$$\begin{aligned} f_n(x + \tau) &= \cos\Big(\frac{2\pi n(x + \tau)}{\tau}\Big),\\ &= \cos\Big(\frac{2\pi n x}{\tau} + 2\pi n\Big),\\ &= \cos\Big(\frac{2\pi n x}{\tau}\Big),\\ &= f_n(x) \quad \Rightarrow \quad f_n(x_\tau) = f_n(x). \end{aligned}$$

A similar calculation shows that g_n is τ-periodic. ◁

It is simple to see that a linear combination of τ-periodic functions is also τ-periodic. Indeed, let f and g be τ-periodic function, that is, $f(x-\tau) = f(x)$ and $g(x-\tau) = g(x)$. Then, given any constants a, b holds

$$a\,f(x-\tau) + b\,g(x-\tau) = a\,f(x) + b\,g(x).$$

A simple application of this result is given in the following example.

Example 7.2.3: Show that the following function is τ-periodic,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{2\pi n x}{\tau}\Big), + b_n \sin\Big(\frac{2\pi n x}{\tau}\Big)\Big].$$

Solution: f is τ-periodic, since it is a linear combination of τ-periodic functions. $\lhd$

The set of infinitely many differentiable functions $f : [-\ell, \ell] \subset \mathbb{R} \to \mathbb{R}$, with $\ell > 0$, together with the operation of linear combination of functions define a vector space. The main difference between this vector space and $\mathbb{R}^n$, for any $n \geqslant 1$, is that the space of functions is infinite dimensional. An inner product in a vector space is an operation that associates to every pair of vectors a real number, and this operation is positive definite, symmetric and bilinear. An inner product in the vector space of functions is defined as follows: Given any two functions f and g, define its inner product, denoted by (f, g), as the following number,

$$(f, g) = \int_{-\ell}^{\ell} f(x)\,g(x)\,dx.$$

This is an inner product since it is positive definite,

$$(f, f) = \int_{-\ell}^{\ell} f^2(x)\,dx \geqslant 0, \quad \text{with} \quad \Big\{ (f, f) = 0 \quad \Leftrightarrow \quad f = 0 \Big\};$$

it is symmetric,

$$(f, g) = \int_{-\ell}^{\ell} f(x)\,g(x)\,dx = \int_{-\ell}^{\ell} g(x)\,f(x)\,dx = (g, f);$$

and bilinear since it is symmetric and linear,

$$\begin{aligned} \big(f, [ag + bh]\big) &= \int_{-\ell}^{\ell} f(x)\,\big[a\,g(x) + b\,h(x)\big]\,dx \\ &= a \int_{-\ell}^{\ell} f(x)\,g(x)\,dx + b \in_{-\ell}^{\ell} f(x)\,h(x)\,dx \\ &= a(f, g) + b(f, h). \end{aligned}$$

An inner product provides the notion of angle in a vector space, and so the notion of orthogonality of vectors. The idea of perpendicular vectors in the three dimensional space is indeed a notion that belongs to a vector space with an inner product, hence it can be generalized to the space of functions. The following result states that certain sine and cosine functions can be perpendicular to each oder.

Theorem 7.2.2 (Orthogonality). *The following relations hold for all $n, m \in \mathbb{N}$,*

$$\int_{-\ell}^{\ell} \cos\Big(\frac{n\pi x}{\ell}\Big)\, \cos\Big(\frac{m\pi x}{\ell}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ \ell & n = m \neq 0, \\ 2\ell & n = m = 0, \end{cases}$$

$$\int_{-\ell}^{\ell} \sin\Big(\frac{n\pi x}{\ell}\Big)\, \sin\Big(\frac{m\pi x}{\ell}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ \ell & n = m, \end{cases}$$

$$\int_{-\ell}^{\ell} \cos\Big(\frac{n\pi x}{\ell}\Big)\, \sin\Big(\frac{m\pi x}{\ell}\Big)\, dx = 0.$$

REMARK: The proof of this result is based in the following trigonometric identities:

$$\cos(\theta)\, \cos(\phi) = \frac{1}{2}\big[\cos(\theta + \phi) + \cos(\theta - \phi)\big],$$

$$\sin(\theta)\, \sin(\phi) = \frac{1}{2}\big[\cos(\theta - \phi) - \cos(\theta + \phi)\big],$$

$$\sin(\theta)\, \cos(\phi) = \frac{1}{2}\big[\sin(\theta + \phi) + \sin(\theta - \phi)\big].$$

Proof of Theorem 7.2.2: We show the proof of the first equation in Theorem 7.2.2, the proof of the other two equations is similar. So, From the trigonometric identities above we obtain

$$\int_{-\ell}^{\ell} \cos\Big(\frac{n\pi x}{\ell}\Big)\, \cos\Big(\frac{m\pi x}{\ell}\Big)\, dx = \frac{1}{2}\int_{-\ell}^{\ell} \cos\Big[\frac{(n+m)\pi x}{\ell}\Big]\, dx + \frac{1}{2}\int_{-\ell}^{\ell} \cos\Big[\frac{(n-m)\pi x}{\ell}\Big]\, dx.$$

Now, consider the case that at least one of n or m is strictly greater than zero. In this case it holds the first term always vanishes, since

$$\frac{1}{2}\int_{-\ell}^{\ell} \cos\Big[\frac{(n+m)\pi x}{\ell}\Big]\, dx = \frac{\ell}{2(n+m)\pi}\, \sin\Big[\frac{(n+m)\pi x}{\ell}\Big]\Big|_{-\ell}^{\ell} = 0;$$

while the remaining term is zero in the sub-case $n \neq m$, due to the same argument as above,

$$\frac{1}{2}\int_{-\ell}^{\ell} \cos\Big[\frac{(n-m)\pi x}{\ell}\Big]\, dx = \frac{\ell}{2(n-m)\pi}\, \sin\Big[\frac{(n-m)\pi x}{\ell}\Big]\Big|_{-\ell}^{\ell} = 0;$$

while in the sub-case that $n = m \neq 0$ we have that

$$\frac{1}{2}\int_{-\ell}^{\ell} \cos\Big[\frac{(n-m)\pi x}{\ell}\Big]\, dx = \frac{1}{2}\int_{-\ell}^{\ell} dx = \ell.$$

Finally, in the case that both $n = m = 0$ is simple to see that

$$\int_{-\ell}^{\ell} \cos\Big(\frac{n\pi x}{\ell}\Big)\, \cos\Big(\frac{m\pi x}{\ell}\Big)\, dx = \int_{-\ell}^{\ell} dx = 2\ell.$$

This establishes the first equation in Theorem 7.2.2. The remaining equations are proven in a similar way. □

The main result of this Section is that any twice continuously differentiable function defined on an interval $[-\ell, \ell] \subset \mathbb{R}$, with $\ell > 0$, admits a Fourier series expansion.

Theorem 7.2.3 (Fourier Series). *Given $\ell > 0$, assume that the functions f, f' and $f'' : [-\ell, \ell] \subset \mathbb{R} \to \mathbb{R}$ are continuous. Then, f can be expressed as an infinite series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big[a_n\, \cos\Big(\frac{n\pi x}{\ell}\Big) + b_n\, \sin\Big(\frac{n\pi x}{\ell}\Big)\Big] \qquad (7.2.4)$$

with the constants a_n and b_n given by

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\cos\Big(\frac{n\pi x}{\ell}\Big)\,dx, \qquad n \geqslant 0, \tag{7.2.5}$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\sin\Big(\frac{n\pi x}{\ell}\Big)\,dx, \qquad n \geqslant 1. \tag{7.2.6}$$

Furthermore, the Fourier series in Eq. (7.2.4) provides a 2ℓ-periodic extension of f from the domain $[-\ell,\ell] \subset \mathbb{R}$ to $\mathbb{R}$.

Remark: The Fourier series given in Eq. (7.2.4) also exists for functions f which are only piecewise continuous, although the proof of the Theorem for such functions is more involved than in the case where f and its first two derivatives are continuous. In this notes we present the simpler proof.

Proof of Theorem 7.2.3: We split the proof in two parts: First, we show that if f admits an infinite series of the form given in Eq. (7.2.4), then the coefficients a_n and b_n must be given by Eqs. (7.2.5)-(7.2.6); second, we show that the functions f_N approaches f as $N \to \infty$, where

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\Big[a_n\,\cos\Big(\frac{n\pi x}{\ell}\Big) + b_n\,\sin\Big(\frac{n\pi x}{\ell}\Big)\Big].$$

Regarding the first part, assume that f can be expressed by a series as given in Eq. (7.2.4). Then, multiply both sides of this equation by a cosine function and integrate as follows,

$$\begin{aligned}\int_{-\ell}^{\ell} f(x)\cos\Big(\frac{m\pi x}{\ell}\Big)\,dx &= \frac{a_0}{2}\int_{-\ell}^{\ell}\cos\Big(\frac{m\pi x}{\ell}\Big)\,dx\\ &\quad+\sum_{n=1}^{\infty}\Big[a_n\int_{-\ell}^{\ell}\cos\Big(\frac{m\pi x}{\ell}\Big)\,\cos\Big(\frac{n\pi x}{\ell}\Big)\,dx\\ &\quad+b_n\int_{-\ell}^{\ell}\cos\Big(\frac{m\pi x}{\ell}\Big)\,\sin\Big(\frac{n\pi x}{\ell}\Big)\,dx\Big].\end{aligned}$$

In the case that $m = 0$ only the first term on the right-hand side above is nonzero, and Theorem 7.2.2 implies that

$$\int_{-\ell}^{\ell} f(x)\,dx = \frac{a_0}{2}\,2\ell \quad\Rightarrow\quad a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx,$$

which agrees with Eq. (7.2.5) for $n = 0$. In the case $m \geqslant 1$ Theorem 7.2.2 implies that

$$\int_{-\ell}^{\ell} f(x)\cos\Big(\frac{m\pi x}{\ell}\Big)\,dx = a_n\ell,$$

which again agrees with Eq. (7.2.5). Instead of multiplying by a cosine one multiplies the equation above by $\sin\big(\frac{n\pi x}{\ell}\big)$, then one obtains Eq. (7.2.6). The second part of the proof is similar and it is left as an exercise. This establishes the Theorem. □

Example 7.2.4: Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 + x & x \in [-1, 0),\\ 1 - x & x \in [0, 1].\end{cases}$$

Solution: The Fourier series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big[a_n\cos(n\pi x) + b_n\sin(n\pi x)\big],$$

where the coefficients a_n, b_n are given in Theorem 7.2.3. We start computing a_0, that is,

$$
\begin{aligned}
a_0 &= \int_{-1}^{1} f(x)\,dx \\
&= \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx \\
&= \Big(x + \frac{x^2}{2}\Big)\Big|_{-1}^{0} + \Big(x - \frac{x^2}{2}\Big)\Big|_{0}^{1} \\
&= \Big(1 - \frac{1}{2}\Big) + \Big(1 - \frac{1}{2}\Big) \quad \Rightarrow \quad a_0 = 1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
a_n &= \int_{-1}^{1} f(x)\,\cos(n\pi x)\,dx \\
&= \int_{-1}^{0} (1+x)\,\cos(n\pi x)\,dx + \int_{0}^{1} (1-x)\,\cos(n\pi x)\,dx.
\end{aligned}
$$

Recalling the integrals

$$
\begin{aligned}
\int \cos(n\pi x)\,dx &= \frac{1}{n\pi}\,\sin(n\pi x), \\
\int x\,\cos(n\pi x)\,dx &= \frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x),
\end{aligned}
$$

it is not difficult to see that

$$
\begin{aligned}
a_n &= \frac{1}{n\pi}\,\sin(n\pi x)\Big|_{-1}^{0} + \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{-1}^{0} \\
&\quad + \frac{1}{n\pi}\,\sin(n\pi x)\Big|_{0}^{1} - \Big[\frac{x}{n\pi}\,\sin(n\pi x) + \frac{1}{n^2\pi^2}\,\cos(n\pi x)\Big]\Big|_{0}^{1} \\
&= \Big[\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}\cos(-n\pi)\Big] - \Big[\frac{1}{n^2\pi^2}\cos(-n\pi) - \frac{1}{n^2\pi^2}\Big],
\end{aligned}
$$

we then conclude that

$$
a_n = \frac{2}{n^2\pi^2}\big[1 - \cos(-n\pi)\big].
$$

Finally, we must find the coefficients b_n. The calculation is similar to the one done above for a_n, and it is left as an exercise: Show that $b_n = 0$. Then, the Fourier series of f is

$$
f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\big[1 - \cos(-n\pi)\big]\,\cos(n\pi x).
$$

$\lhd$

7.2.3. **Even and odd functions.** There exist particular classes of functions with simple Fourier series expansions. Simple means that either the b_n or the a_n coefficients vanish. These functions are called even or odd, respectively.

Definition 7.2.4. *Given any $\ell > 0$, a function $f : [-\ell, \ell\,] \to \mathbb{R}$ is called **even** iff holds*

$$
f(-x) = f(x), \quad \textit{for all} \quad x \in [-\ell, \ell\,].
$$

*A function $f : [-\ell, \ell\,] \to \mathbb{R}$ is called **odd** iff holds*

$$
f(-x) = -f(x), \quad \textit{for all} \quad x \in [-\ell, \ell\,].
$$

Example 7.2.5: Even functions are the following:

$$f(x) = \cos(x), \qquad f(x) = x^2, \qquad f(x) = 3x^4 - 7x^2 + \cos(3x).$$

Odd functions are the following:

$$f(x) = \sin(x), \qquad f(x) = x^3, \qquad f(x) = 3x^3 - 7x + \sin(3x).$$

There exist functions that are neither even nor odd:

$$f(x) = e^x, \qquad f(x) = x^2 + 2x - 1.$$

Notice that the product of two odd functions is even: For example $f(x) = x\sin(x)$ is even, while both x and $\sin(x)$ are odd functions. Also notice that the product of an odd function with an even function is again an odd function. ◁

The Fourier series of a function which is either even or odd is simple to find.

Theorem 7.2.5 (Cosine and sine series). *Consider the Fourier series of the function $f : [-\ell, \ell] \to \mathbb{R}$, that is,*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{\ell}\Big) + b_n \sin\Big(\frac{n\pi x}{\ell}\Big) \Big].$$

(a) *The function f is even iff the coefficients $b_n = 0$ for all $n \geqslant 1$. In this case the Fourier series is called a* ***cosine*** *series.*
(b) *The function f is odd iff the coefficients $a_n = 0$ for all $n \geqslant 0$. In this case the Fourier series is called a* ***sine*** *series.*

Proof of Theorem 7.2.5:

Part (a): Suppose that f is even, that is, $f(-x) = f(x)$, and compute the coefficients b_n,

$$\begin{aligned}
b_n &= \int_{-\ell}^{\ell} f(x)\, \sin\Big(\frac{n\pi x}{\ell}\Big)\, dx \\
&= \int_{\ell}^{-\ell} f(-y)\, \sin\Big(\frac{n\pi(-y)}{\ell}\Big)\, (-dy), \quad y = -x, \quad dy = -dx, \\
&= \int_{-\ell}^{\ell} f(-y)\, \sin\Big(\frac{n\pi(-y)}{\ell}\Big)\, dy, \\
&= -\int_{-\ell}^{\ell} f(y)\, \sin\Big(\frac{n\pi y}{\ell}\Big)\, dy, \\
&= -b_n \quad \Rightarrow \quad b_n = 0.
\end{aligned}$$

Part (b): The proof is similar. This establishes the Theorem. □

7.2.4. **Exercises.**

7.2.1.- .

7.2.2.- .

7.3. The Heat Equation

We now solve our first *partial* differential equation, the heat equation, which describes the temperature of a material as function of time and space. The equation contains partial derivatives of both time and space variables. We solve this equation using the separation of variables method, which transforms the partial differential equation into a set of infinitely many ordinary differential equations.

7.3.1. The Initial-Boundary Value Problem.

Consider a solid bar as the one sketched in Fig. 45. Let u be the temperature in that bar. Assume that u depends on time t and only one space coordinate x, so the temperature is a function with values $u(t,x)$. This assumption simplifies the mathematical problem we are about to solve. This is not an unreal assumption, this situation exists in the real world. One needs to thermally insulate all horizontal surfaces of the bar and provide initial and boundary conditions that do not depend on neither y or z. Anyway, besides assuming that u depends only on t and x, we also assume that the temperature of the bar is held constant on the surfaces $x = 0$ and $x = \ell$, with values $u(t,0) = 0$ and $u(t,\ell) = 0$. See Fig. 45.

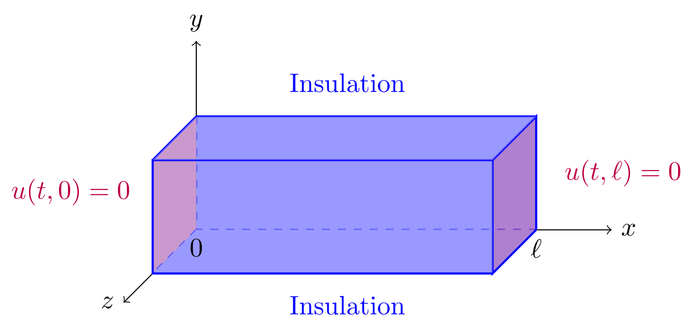


Figure 45. A solid bar thermally insulated on all surfaces except the $x = 0$ and $x = \ell$ surfaces, which are held at temperatures T_0 and T_ℓ, respectively. The temperature u of the bar is a function of the coordinate x only.

The one-space dimensional heat equation for the temperature function u is the partial differential equation

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x),$$

where ∂ denotes partial derivative and k is a positive constant called the thermal diffusivity of the material. This equation has infinitely many solutions. The temperature of the bar is given by a uniquely defined function u because this function satisfies a few extra conditions. We mentioned the boundary conditions at $x = 0$ and $x = \ell$. We also need to specify the initial temeperature of the bar. The partial differential equation and these extra conditions define an initial-boundary value problem, which we now summarize.

Definition 7.3.1. *The **Initial-Boundary Value Problem** for the one-space dimensional heat equation with homogeneous boundary conditions is the following: Given positive constants ℓ and k, and a function $f : [0,\ell] \to \mathbb{R}$ satisfying $f(0) = f(\ell) = 0$, find a function $u : [0,\infty) \times [0,\ell] \to \mathbb{R}$, with values $u(t,x)$, solution of*

$$\partial_t u(t,x) = k\,\partial_x^2 u(t,x), \tag{7.3.1}$$

$$u(0,x) = f(x), \tag{7.3.2}$$

$$u(t,0) = 0, \qquad u(t,\ell) = 0. \tag{7.3.3}$$

The requirement in Eq. (7.3.2) is called an initial condition, while the equations in (7.3.3) are called the boundary conditions of the problem. The latter conditions are actually homogeneous boundary conditions. A sketch on the tx plane is useful to understand this type of problems, as can be seen in Fig. 46. This figure helps us realize that the boundary conditions $u(t,0) = u(t,\ell) = 0$ hold for all times $t > 0$. And this figure also helps understand why the initial condition function f must satisfy the compatibility conditions $f(0) = f(\ell) = 0$.

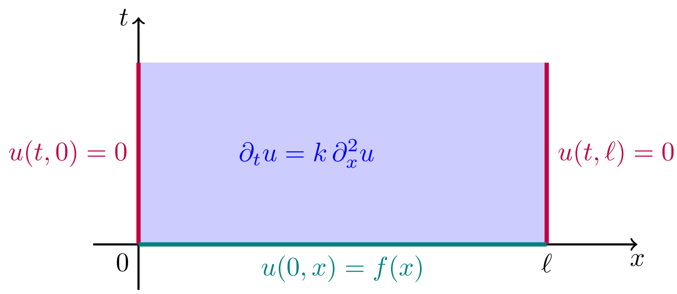


Figure 46. A sketch on the tx plane of an initial-boundary value problem for the heat equation.

I think it was Richard Feynman who said that one should never start a calculation without knowing the answer. Following that advice we now try to understand the qualitative behavior of a solution to the heat equation. Suppose that the boundary conditions are $u(t,0) = T_0 = 0$ and $u(t,\ell) = T_\ell > 0$. Suppose that at a fixed time $t \geqslant 0$ the graph of the temperature function u is as in Fig. 47. Then a qualitative idea of how a solution of the heat equation behaves can be obtained from the arrows in that figure. The heat equation relates the time variation of the temperature, $\partial_t u$, to the curvature of the function u in the x variable, $\partial_x^2 u$. In the regions where the function u is concave up, hence $\partial_x^2 u > 0$, the heat equation says that the tempreature must increase $\partial_t u > 0$. In the regions where the function u is concave down, hence $\partial_x^2 u < 0$, the heat equation says that the tempreature must decrease $\partial_t u < 0$. So the heat equation tries to make the temperature along the material to vary the least possible that is consistent with the boundary condition. In the case of the Figure, the temperature will try to get to the dashed line.

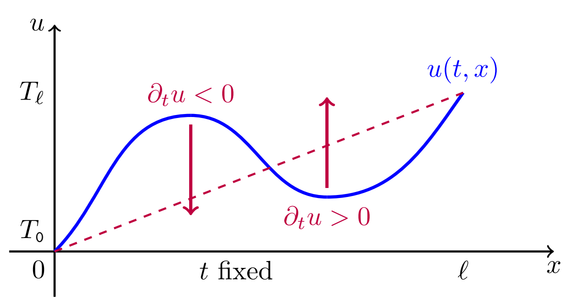


Figure 47. Qualitative behavior of a solution to the heat equation.

We now summarize the main result about the initial-boundary value problem.

Theorem 7.3.2. *If the initial data function f is continuous, then the initial boundary value problem given in Def. 7.3.1 has a unique solution u given by*

$$u(t,x) = \sum_{n=1}^{\infty} c_n \, e^{-(\frac{n\pi}{\ell})^2 t} \sin\Big(\frac{n\pi x}{\ell}\Big), \tag{7.3.4}$$

where the coefficients c_n are given in terms of the initial data

$$c_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\Big(\frac{n\pi x}{\ell}\Big)\, dx.$$

Remarks:

(a) The theorem considers only homogeneous boundary conditions. The analysis given in Fig. 47 predicts that the temperature will drop to zero, to match the boundary values. This is what we see in the solution formula in Eq. (7.3.4), which says that the temperature approaches zero exponentially in time.
(b) Each term in the infinite sum in Eq. (7.3.4) satisfies the boundary conditions, because of the factor with the sine function.
(c) The solution formula evaluated at $t = 0$ is the Sine Fourier series expansion of the initial data function f, as can be seen by the formula for the coefficient c_n.
(d) The proof of this theorem is based in the separation of variables method and is presented in the next subsection.

7.3.2. **The Separation of Variables.** We present two versions of the same proof of Theorem 7.3.2. They differ only in the emphasis on different parts of the argument.

First Proof of Theorem 7.3.2: The separation of variables method it is usually presented in the literature as follows. First look for particular type of solutions to the heat equation of the form

$$u(t,x) = v(t)\, w(x).$$

Introducing this particular function in the heat equation we get

$$\dot v(t)\, w(x) = k\, v(t)\, w''(x) \quad \Rightarrow \quad \frac{1}{k}\,\frac{\dot v(t)}{v(t)} = \frac{w''(x)}{w(x)},$$

where we used the notation $\dot v = dv/dt$ and $w' = dw/dx$. These equations are the reason the method is called separation of variables. The left hand side in the last equation above depends only on t and the right hand side depends only on x. The only possible solution is that both sides are equal the same constant, call it $-\lambda$. So we end up with two equations

$$\frac{1}{k}\,\frac{\dot v(t)}{v(t)} = -\lambda, \qquad \frac{w''(x)}{w(x)} = -\lambda.$$

The first equation leads to an initial value problem for v once initial conditions are provided. The second equation leads to an eigenvalue-eigenfunction problem for w once boundary conditions are provided. The choice of these inital and boundary conditions is inspired from the analogous conditions in Def. 7.3.1. Usually in the literature these conditions are

$$v(0) = 1, \quad \text{and} \quad w(0) = w(\ell) = 0.$$

The boundary conditions on w are clearly coming from the boundary conditions in Def 7.3.1, but the initial condition on v is clearly not. We now solve the eigenvalue-eigenfunction problem for w, and we know from § 7.1 that the solution is

$$\lambda_n = \Big(\frac{n\pi}{\ell}\Big)^2, \quad w_n(x) = \sin\Big(\frac{n\pi x}{\ell}\Big), \quad n = 1, 2, \cdots .$$

The solution for the initial value problem is

$$v(t) = e^{-k(\frac{n\pi}{\ell})^2 t},$$

so we got a particular solution

$$u_n(t,x) = e^{-k(\frac{n\pi}{\ell})^2 t} \sin\Big(\frac{n\pi x}{\ell}\Big),$$

where $n = 1, 2, \cdots$. Then any linear combination of these solutions is also a solution of the heat equation, so the function

$$u(t,x) = \sum_{n=1}^{\infty} c_n\, e^{-k(\frac{n\pi}{\ell})^2 t} \sin\Big(\frac{n\pi x}{\ell}\Big)$$

is solution of the heat equation and satisfies the homogeneous boundary conditions because each term in that sum does. We now want that the function u be solution of the initial-boundary value problem in Def. 7.3.1. Then at $t = 0$ we get the equation

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\Big(\frac{n\pi x}{\ell}\Big).$$

This is the Sine Fourier series expansion of the initial data function, so the orthogonality of the sine functions implies

$$c_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\Big(\frac{n\pi x}{\ell}\Big)\, dx.$$

This establishes the Theorem. □

The second proof emphasises more the vector space aspects of the problem.

Second Proof of Theorem 7.3.2: Consider the vector space

$$V = \{v,\ \text{differentiable functions on } [0,\ell],\ \text{with } v(0) = v(\ell) = 0\}.$$

Introduce in this vector space the following inner product. Given two functions $f, g \in V$, the inner product of these two functions is defined as

$$f \cdot g = \int_0^{\ell} f(x)\, g(x)\, dx.$$

Let the set $\{w_n\}_{n=1}^{\infty}$ be a basis of the vector space V. In particular, these functions w_n satisfy the boundary conditions $w_n(0) = w_n(\ell) = 0$. At this point we do not know the explicit expression for this basis. Now, if u is a solution to the initial-boundary value problem for the heat equation, that solution u at each value of t determines an element in the space V. So we can write it in terms of the basis vectors,

$$u(t,x) = \sum_{n=1}^{\infty} v_n(t)\, w_n(x),$$

where the components of the function u in the basis w_n, the coefficients v_n, are actually functions of t. This function u is solution of the heat equation iff holds

$$\sum_{n=1}^{\infty} \big[\partial_t(v_n\, w_n) - k\, \partial_x^2(v_n\, w_n)\big] = 0.$$

A sufficient condition for the sum above to vanish is that each term vanishes,

$$\partial_t\big(v_n w_n\big) = k\, \partial_x^2\big(v_n w_n\big).$$

Since v_n depends only on t and w_n depend only on x, we get the equations

$$\dot v_n(t)\, w_n(x) = k\, v_n(t)\, w_n''(x) \quad \Rightarrow \quad \frac{1}{k}\, \frac{\dot v_n(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)},$$

where we used the notation $\dot{v}_n = dv_n/dt$ and $w_n' = dw_n/dx$. As we said in the first proof above, these equations are the reason the method is called separation of variables. The left hand side in the last equation above depends only on t and the right hand side depends only on x. The only possible solution is that both sides are equal the same constant, call it $-\lambda_n$. So we end up with the equations

$$\frac{1}{k}\,\frac{\dot{v}_n(t)}{v_n(t)} = -\lambda_n, \qquad \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

Recall that the basis vectors w_n satisfy the boundary conditions $w_n(0) = w_n(\ell) = 0$. This is an eigenvalue-eigenfunction problem, which we solved in § 7.1. The result is

$$\lambda_n = \Big(\frac{n\pi}{\ell}\Big)^2, \quad w_n(x) = \sin\Big(\frac{n\pi x}{\ell}\Big), \quad n = 1, 2, \cdots .$$

Using the value of λ_n found above, the solution for the fuunction v_n is

$$v(t) = v_n(0)\, e^{-k(\frac{n\pi}{\ell})^2 t}.$$

So we have a solution to the heat equation given by

$$u(t,x) = \sum_{n=1}^{\infty} v_n(0)\, e^{-k(\frac{n\pi}{\ell})^2 t} \sin\Big(\frac{n\pi x}{\ell}\Big).$$

This solution satisfied the boundary conditions $u(t,0) = u(t,\ell) = 0$ because each term satisfy them. The constants $v_n(0)$ are determined from the initial data,

$$f(x) = \sum_{n=1}^{\infty} v_n(0)\, \sin\Big(\frac{n\pi x}{\ell}\Big).$$

Recall now that the sine functions above are mutually orthogonal and that

$$\int_0^{\ell} \sin\Big(\frac{n\pi x}{\ell}\Big)\, \sin\Big(\frac{m\pi x}{\ell}\Big)\, dx = \begin{cases} 0 & n \neq m, \\ \dfrac{\ell}{2} & n = m, \end{cases}$$

Then, multiplying the equation for f by a $sin(n\pi x/\ell)$ and integrating on $[0,\ell]$ it is not so difficult to get

$$v_n(0) = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\Big(\frac{n\pi x}{\ell}\Big)\, dx.$$

This establishes the Theorem. □

Example 7.3.1: Find the solution to the initial-boundary value problem

$$4\,\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$$

with initial and boundary conditions given by

$$u(0,x) = 3\sin(\pi x/2), \quad u(t,0) = 0, \quad u(t,2) = 0.$$

Solution: We write the solution as a series expansion

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t,x), \quad \text{where} \quad u_n(t,x) = v_n(t)\, w_n(x).$$

Then a sufficient condition for u to solve the heat equation is

$$4w_n(x)\,\frac{dv}{dt}(t) = v_n(t)\,\frac{d^2w}{dx^2}(x) \quad \Rightarrow \quad \frac{4\dot{v}_n(t)}{v_n(t)} = \frac{w_n''(x)}{w_n(x)} = -\lambda_n.$$

The equations for v_n and w_n are

$$\dot v_n(t) + \frac{\lambda_n}{4}\, v_n(t) = 0, \qquad w_n''(x) + \lambda_n\, w_n(x) = 0.$$

We solve for v_n using the integrating factor method, as in § 1.1,

$$e^{\frac{\lambda_n}{4}t}\, \dot v_n(t) + \frac{\lambda_n}{4}\, e^{\frac{\lambda_n}{4}t}\, v_n(t) = 0 \quad \Rightarrow \quad \frac{d}{dt}\big[e^{\frac{\lambda_n}{4}t}\, v_n\big] = 0.$$

Therefore, we get

$$v_n(t) = v_n(0)\, e^{-\frac{\lambda_n}{4}t},$$

Next we turn to the boundary value problem for w_n. We need to find the solution of

$$w_n''(x) + \lambda_n\, w_n(x) = 0, \quad \text{with} \quad w_n(0) = w_n(2) = 0.$$

This is an eigenvalue-eigenfunction problem for w_n and λ_n. From § 7.1 we know that this problem has solutions only for $\lambda_n > 0$. Following the calculations in that section we introduce $\lambda_n = \mu_n^2$. The characteristic polynomial of the differential equation is

$$p(r) = r^2 + \mu_n^2 = 0 \quad \Rightarrow \quad r_{n\pm} = \pm\mu_n i.$$

The general solution of the differential equation is

$$w_n(x) = c_1\, \cos(\mu_n x) + c_2\, \sin(\mu_n x).$$

The first boundary conditions on w_n implies

$$0 = w_n(0) = c_1, \quad \Rightarrow \quad w_n(x) = c_2\, \sin(\mu_n x).$$

The second boundary condition on w_n implies

$$0 = w_n(2) = c_2\, \sin(\mu_n 2), \quad c_2 \neq 0, \quad \Rightarrow \quad \sin(\mu_n 2) = 0.$$

Then, $\mu_n 2 = n\pi$, that is, $\mu_n = \dfrac{n\pi}{2}$. Choosing $c_2 = 1$, we conclude,

$$\lambda_m = \Big(\frac{n\pi}{2}\Big)^2, \qquad w_n(x) = \sin\Big(\frac{n\pi x}{2}\Big), \quad n = 1, 2, \cdots.$$

The expressions for v_n and w_n imply that the solution u has the form

$$u(t,x) = \sum_{n=1}^{\infty} v_n(0)\, e^{-(\frac{n\pi}{4})^2 t}\, \sin\Big(\frac{n\pi x}{2}\Big).$$

The initial condition is

$$3\, \sin\Big(\frac{\pi x}{2}\Big) = \sum_{n=1}^{\infty} v_n(0)\, \sin\Big(\frac{n\pi x}{2}\Big).$$

The orthogonality of the sine functions above implies

$$3\int_0^2 \sin\Big(\frac{\pi x}{2}\Big)\, \sin\Big(\frac{m\pi x}{2}\Big)\, dx = \sum_{n=1}^{\infty} v_n(0) \int_0^2 \sin\Big(\frac{n\pi x}{2}\Big)\, \sin\Big(\frac{m\pi x}{2}\Big)\, dx.$$

If $m \neq 1$, then $0 = v_m(0)\, \frac{2}{2}$, that is, $v_m(0) = 0$ for $m \neq 1$. Therefore we get,

$$3 \sin\Big(\frac{\pi x}{2}\Big) = v_1(0) \sin\Big(\frac{\pi x}{2}\Big) \quad \Rightarrow \quad v_1(0) = 3.$$

So the solution of the initial-boundary value problem for the heat equation is

$$u(t,x) = 3\, e^{-(\frac{\pi}{4})^2 t}\, \sin\Big(\frac{\pi x}{2}\Big).$$

◁

7.3.3. Exercises.

7.3.1.- .

7.3.2.- .

Chapter 8. Review of Linear Algebra

8.1. Systems of Algebraic Equations

We said in the previous Section that one way to solve a linear differential system is to transform the original system and unknowns into a decoupled system, solve the decoupled system for the new unknowns, and then transform back the solution to the original unknowns. These transformations, to and from, the decoupled system are the key steps to solve a linear differential system. One way to understand these transformations is using the ideas and notation from Linear Algebra. This Section is a review of concepts from Linear Algebra needed to introduce the transformations mentioned above. We start introducing an algebraic linear system, we then introduce matrices and matrix operations, column vectors and matrix vector products. The transformations on a system of differential equations mentioned above will be introduced in a later Section, when we study the eigenvalues and eigenvectors of a square matrix.

8.1.1. **Linear algebraic systems.** One could say that the set of results we call Linear Algebra originated with the study of linear systems of algebraic equations. Our review of elementary concepts from Linear Algebra starts with a study of these type of equations.

Definition 8.1.1. *An $n \times n$ **system of linear algebraic equations** is the following: Given constants a_{ij} and b_i, where $i, j = 1 \cdots, n \geqslant 1$, find the constants x_j solutions of*

$$a_{11} x_1 + \cdots + a_{1n} x_n = b_1, \tag{8.1.1}$$

$$\vdots$$

$$a_{n1} x_1 + \cdots + a_{nn} x_n = b_n. \tag{8.1.2}$$

*The system is called **homogeneous** iff all sources vanish, that is, $b_1 = \cdots = b_n = 0$.*

Example 8.1.1:

(a) A 2×2 linear system on the unknowns x_1 and x_2 is the following:

$$\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3. \end{aligned}$$

(b) A 3×3 linear system on the unknowns x_1, x_2 and x_3 is the following:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}$$

◁

One way to find a solution to an $n \times n$ linear system is by substitution. Compute x_1 from the first equation and introduce it into all the other equations. Then compute x_2 from this new second equation and introduce it into all the remaining equations. Repeat this procedure till the last equation, where one finally obtains x_n. Then substitute back and find all the x_i, for $i = 1, \cdots, n-1$. A computational more efficient way to find a solution is to perform Gauss elimination operations on the augmented matrix of the system. Since matrix notation will simplify calculations, it is convenient we spend some time on this. We start with the basic definitions.

Definition 8.1.2. *An $m \times n$ **matrix**, A, is an array of numbers*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{matrix} m \text{ rows}, \\ n \text{ columns}, \end{matrix} \qquad a_{ij} \in \mathbb{C},$$

*where $i = 1, \cdots, m$, and $j = 1, \cdots, n$. An $n \times n$ matrix is called a **square matrix**.*

EXAMPLE 8.1.2:

(a) Examples of 2×2, 2×3, 3×2 real-valued matrices, and a 2×2 complex-valued matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \qquad D = \begin{bmatrix} 1+i & 2-i \\ 3 & 4i \end{bmatrix}.$$

(b) The coefficients of the algebraic linear systems in Example 8.1.1 can be grouped in matrices, as follows,

$$\left.\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3, \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \qquad \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}\right\} \Rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix}.$$

◁

The particular case of an $m \times 1$ matrix is called an m-vector.

Definition 8.1.3. *An **m-vector**, $\boldsymbol{v}$, is the array of numbers $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$, where the vector components $v_i \in \mathbb{C}$, with $i = 1, \cdots, m$.*

EXAMPLE 8.1.3: The unknowns of the algebraic linear systems in Example 8.1.1 can be grouped in vectors, as follows,

$$\left.\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3, \end{aligned}\right\} \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad \left.\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ -3x_1 + x_2 + 3x_3 &= 24, \\ x_2 - 4x_3 &= -1. \end{aligned}\right\} \Rightarrow \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

◁

Definition 8.1.4. *The **matrix-vector product** of an $n \times n$ matrix A and an n-vector $\boldsymbol{x}$ is an n-vector given by*

$$A\boldsymbol{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{1n}x_n \end{bmatrix}$$

The matrix-vector product of an $n \times n$ matrix with an n-vector is another n-vector. This product is useful to express linear systems of algebraic equations in terms of matrices and vectors.

EXAMPLE 8.1.4: Find the matrix-vector products for the matrices A and vectors $\boldsymbol{x}$ in Examples 8.1.2(b) and Example 8.1.3, respectively.

SOLUTION: In the 2×2 case we get

$$A\boldsymbol{x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix}.$$

In the 3×3 case we get,

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -3x_1 + x_2 + 3x_3 \\ x_2 - 4x_3 \end{bmatrix}.$$

◁

Example 8.1.5: Use the matrix-vector product to express the algebraic linear system below,

$$2x_1 - x_2 = 0,$$
$$-x_1 + 2x_2 = 3.$$

Solution: Introduce the coefficient matrix A, the unknown vector $\boldsymbol{x}$, and the source vector $\boldsymbol{b}$ as follows,

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Since the matrix-vector product $A\boldsymbol{x}$ is given by

$$A\boldsymbol{x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix},$$

then we conclude that

$$\left.\begin{aligned} 2x_1 - x_2 &= 0, \\ -x_1 + 2x_2 &= 3, \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \Leftrightarrow \quad A\boldsymbol{x} = \boldsymbol{b}.$$

◁

It is simple to see that the result found in the Example above can be generalized to every $n \times n$ algebraic linear system.

Theorem 8.1.5. *Given the algebraic linear system in Eqs. (8.1.1)-(8.1.2), introduce the coefficient matrix A, the unknown vector $\boldsymbol{x}$, and the source vector $\boldsymbol{b}$, as follows,*

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Then, the algebraic linear system can be written as

$$A\boldsymbol{x} = \boldsymbol{b}.$$

Proof of Theorem 8.1.5: From the definition of the matrix-vector product we have that

$$A\mathbf{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{1n}x_n \end{bmatrix}.$$

Then, we conclude that

$$\left.\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n, \end{aligned}\right\} \quad \Leftrightarrow \quad \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{1n}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad \Leftrightarrow \quad A\boldsymbol{x} = \boldsymbol{b}.$$

□

We introduce one last definition, which will be helpful in the next subsection.

Definition 8.1.6. *The **augmented matrix** of $A\boldsymbol{x} = \boldsymbol{b}$ is the $n \times (n+1)$ matrix $[A|\boldsymbol{b}]$.*

The augmented matrix of an algebraic linear system contains the equation coefficients and the sources. Therefore, the augmented matrix of a linear system contains the complete information about the system.

EXAMPLE 8.1.6: Find the augmented matrix of both the linear systems in Example 8.1.1.

SOLUTION: The coefficient matrix and source vector of the first system imply that

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad \Rightarrow \quad [A|\boldsymbol{b}] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right].$$

The coefficient matrix and source vector of the second system imply that

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 24 \\ -1 \end{bmatrix} \quad \Rightarrow \quad [A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -3 & 1 & 3 & 24 \\ 0 & 1 & -4 & -1 \end{array}\right].$$

◁

Recall that the linear combination of two vectors is defined component-wise, that is, given any numbers $a, b \in \mathbb{R}$ and any vectors $\boldsymbol{x}$, $\boldsymbol{y}$, their *linear combination* is the vector given by

$$a\boldsymbol{x} + b\boldsymbol{y} = \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix}, \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

With this definition of linear combination of vectors it is simple to see that the matrix-vector product is a linear operation.

Theorem 8.1.7. *The matrix-vector product is a linear operation, that is, given an $n \times n$ matrix A, then for all n-vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and all numbers $a, b \in \mathbb{R}$ holds*

$$A(a\boldsymbol{x} + b\boldsymbol{y}) = a\,A\boldsymbol{x} + b\,A\boldsymbol{y}. \tag{8.1.3}$$

Proof of Theorem 8.1.7: Just write down the matrix-vector product in components,

$$A(a\mathbf{x} + b\mathbf{y}) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix} = \begin{bmatrix} a_{11}(ax_1 + by_1) + \cdots + a_{1n}(ax_n + by_n) \\ \vdots \\ a_{n1}(ax_1 + by_1) + \cdots + a_{nn}(ax_n + by_n) \end{bmatrix}.$$

Expand the linear combinations on each component on the far right-hand side above and re-order terms as follows,

$$A(a\mathbf{x} + b\mathbf{y}) = \begin{bmatrix} a\,(a_{11}x_1 + \cdots + a_{1n}x_n) + b\,(a_{11}y_1 + \cdots + a_{1n}y_n) \\ \vdots \\ a\,(a_{n1}x_1 + \cdots + a_{nn}x_n) + b\,(a_{n1}y_1 + \cdots + a_{nn}y_n) \end{bmatrix}.$$

Separate the right-hand side above,

$$A(a\mathbf{x} + b\mathbf{y}) = a \begin{bmatrix} (a_{11}x_1 + \cdots + a_{1n}x_n) \\ \vdots \\ (a_{n1}x_1 + \cdots + a_{nn}x_n) \end{bmatrix} + b \begin{bmatrix} (a_{11}y_1 + \cdots + a_{1n}y_n) \\ \vdots \\ (a_{n1}y_1 + \cdots + a_{nn}y_n) \end{bmatrix}.$$

We then conclude that

$$A(a\mathbf{x} + b\mathbf{y}) = a\,A\boldsymbol{x} + b\,A\boldsymbol{y}.$$

This establishes the Theorem. □

8.1.2. **Gauss elimination operations.** We now review three operations that can be performed on an augmented matrix of a linear system. These operations change the augmented matrix of the system but they do not change the solutions of the system. The Gauss elimination operations were already known in China around 200 BC. We call them after Carl Friedrich Gauss, since he made them very popular around 1810, when he used them to study the orbit of the asteroid Pallas, giving a systematic method to solve a 6×6 algebraic linear system.

Definition 8.1.8. *The **Gauss elimination operations** are three operations on a matrix:*

(i) Adding to one row a multiple of the another;
(ii) Interchanging two rows;
(iii) Multiplying a row by a non-zero number.

These operations are respectively represented by the symbols given in Fig. 48.

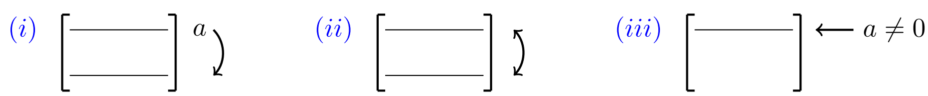


Figure 48. A sketch of the Gauss elimination operations.

As we said above, the Gauss elimination operations change the coefficients of the augmented matrix of a system but do not change its solution. Two systems of linear equations having the same solutions are called *equivalent*. It can be shown that there is an algorithm using these operations that transforms any $n \times n$ linear system into an equivalent system where the solutions are explicitly given.

Example 8.1.7: Find the solution to the 2×2 linear system given in Example 8.1.1 using the Gauss elimination operations.

Solution: Consider the augmented matrix of the 2×2 linear system in Example (8.1.1), and perform the following Gauss elimination operations,

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 1 & 2 \end{array}\right] \to$$

$$\left[\begin{array}{cc|c} 2 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad \Leftrightarrow \quad \left\{\begin{array}{l} x_1 + 0 = 1 \\ 0 + x_2 = 2 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} x_1 = 1 \\ x_2 = 2 \end{array}\right.$$

◁

Example 8.1.8: Find the solution to the 3×3 linear system given in Example 8.1.1 using the Gauss elimination operations

Solution: Consider the augmented matrix of the 3×3 linear system in Example 8.1.1 and perform the following Gauss elimination operations,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -3 & 1 & 3 & 24 \\ 0 & 1 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 7 & 6 & 27 \\ 0 & 1 & -4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & 7 & 6 & 27 \end{array}\right],$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 34 & 34 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 9 & 3 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad \left\{\begin{array}{l} x_1 = -6, \\ x_2 = 3, \\ x_3 = 1. \end{array}\right.$$

◁

In the last augmented matrix on both Examples 8.1.7 and 8.1.8 the solution is given explicitly. This is not always the case with every augmented matrix. A precise way to define the final augmented matrix in the Gauss elimination method is captured in the notion of echelon form and reduced echelon form of a matrix.

Definition 8.1.9. *An $m \times n$ matrix is in* ***echelon form*** *iff the following conditions hold:*

(i) The zero rows are located at the bottom rows of the matrix;
(ii) The first non-zero coefficient on a row is always to the right of the first non-zero coefficient of the row above it.

The ***pivot*** *coefficient is the first non-zero coefficient on every non-zero row in a matrix in echelon form.*

EXAMPLE 8.1.9: The 6×8, 3×5 and 3×3 matrices given below are in echelon form, where the $*$ means any non-zero number and pivots are highlighted.

$$\begin{bmatrix} * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

◁

EXAMPLE 8.1.10: The following matrices are in echelon form, with pivot highlighted.

$$\begin{bmatrix} \mathbf{1} & 3 \\ 0 & \mathbf{1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & 3 & 2 \\ 0 & \mathbf{4} & -2 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{2} & 1 & 1 \\ 0 & \mathbf{3} & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

◁

Definition 8.1.10. *An $m \times n$ matrix is in* ***reduced echelon form*** *iff the matrix is in echelon form and the following two conditions hold:*

(i) The pivot coefficient is equal to 1;
(ii) The pivot coefficient is the only non-zero coefficient in that column.

We denote by E_A a reduced echelon form of a matrix A.

EXAMPLE 8.1.11: The 6×8, 3×5 and 3×3 matrices given below are in echelon form, where the $*$ means any non-zero number and pivots are highlighted.

$$\begin{bmatrix} \mathbf{1} & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & \mathbf{1} & 0 & * & * & 0 & * \\ 0 & 0 & 0 & \mathbf{1} & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & * & 0 & * & * \\ 0 & 0 & \mathbf{1} & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}.$$

◁

EXAMPLE 8.1.12: And the following matrices are not only in echelon form but also in reduced echelon form; again, pivot coefficients are highlighted.

$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & 0 & 4 \\ 0 & \mathbf{1} & 5 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

◁

Summarizing, the Gauss elimination operations can transform any matrix into reduced echelon form. Once the augmented matrix of a linear system is written in reduced echelon form, it is not difficult to decide whether the system has solutions or not.

Example 8.1.13: Use Gauss operations to find the solution of the linear system

$$2x_1 - x_2 = 0,$$
$$-\frac{1}{2}x_1 + \frac{1}{4}x_2 = -\frac{1}{4}.$$

Solution: We find the system augmented matrix and perform appropriate Gauss elimination operations,

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

From the last augmented matrix above we see that the original linear system has the same solutions as the linear system given by

$$2x_1 - x_2 = 0,$$
$$0 = 1.$$

Since the latter system has no solutions, the original system has no solutions. ◁

The situation shown in Example 8.1.13 is true in general. If the augmented matrix $[A|\boldsymbol{b}]$ of an algebraic linear system is transformed by Gauss operations into the augmented matrix $[\tilde{A}|\tilde{\boldsymbol{b}}]$ having a row of the form $[0, \cdots, 0|1]$, then the original algebraic linear system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution.

Example 8.1.14: Find all vectors $\boldsymbol{b}$ such that the system $A\boldsymbol{x} = \boldsymbol{b}$ has solutions, where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

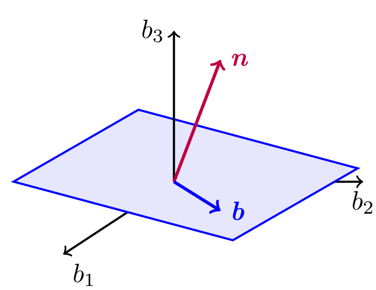
Solution: We do not need to write down the algebraic linear system, we only need its augmented matrix,

$$[A|\boldsymbol{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & -1 & 1 & b_1 + b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 3 & -3 & b_3 - 2b_1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_3 + b_1 + 3b_2 \end{array}\right].$$

Therefore, the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has solutions $\Leftrightarrow$ the source vector satisfies the equation holds $b_1 + 3b_2 + b_3 = 0$.
That is, every source vector $\boldsymbol{b}$ that lie on the plane normal to the vector $\boldsymbol{n}$ is a source vector such that the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has solution, where

$$\boldsymbol{n} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$



◁

8.1.3. **Linearly dependence.** We generalize the idea of two vectors lying on the same line, and three vectors lying on the same plane, to an arbitrary number of vectors.

Definition 8.1.11. *A set of vectors* $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k\}$*, with* $k \geqslant 1$ *is called* ***linearly dependent*** *iff there exists constants* $c_1, \cdots, c_k$*, with at least one of them non-zero, such that*

$$c_1\,\boldsymbol{v}_1 + \cdots + c_k\,\boldsymbol{v}_k = \boldsymbol{0}. \tag{8.1.4}$$

The set of vectors is called ***linearly independent*** *iff it is not linearly dependent, that is, the only constants* $c_1, \cdots, c_k$ *that satisfy Eq. (8.1.4) are given by* $c_1 = \cdots = c_k = 0$.

In other words, a set of vectors is linearly dependent iff one of the vectors is a linear combination of the other vectors. When this is not possible, the set is called linearly independent.

Example 8.1.15: Show that the following set of vectors is linearly dependent,

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\5\end{bmatrix}\right\},$$

and express one of the vectors as a linear combination of the other two.

Solution: We need to find constants c_1, c_2, and c_3 solutions of the equation

$$\begin{bmatrix}1\\2\\3\end{bmatrix} c_1 + \begin{bmatrix}3\\2\\1\end{bmatrix} c_2 + \begin{bmatrix}-1\\2\\5\end{bmatrix} c_3 = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix}1 & 3 & -1\\2 & 2 & 2\\3 & 1 & 5\end{bmatrix} \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The solution to this linear system can be obtained with Gauss elimination operations,

$$\begin{bmatrix}1 & 3 & -1\\2 & 2 & 2\\3 & 1 & 5\end{bmatrix} \to \begin{bmatrix}1 & 3 & -1\\0 & -4 & 4\\0 & -8 & 8\end{bmatrix} \to \begin{bmatrix}1 & 3 & -1\\0 & 1 & -1\\0 & 1 & -1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \quad \Rightarrow \quad \begin{cases} c_1 = -2c_3,\\ c_2 = c_3,\\ c_3 = \text{ free.}\end{cases}$$

Since there are non-zero constants c_1, c_2, c_3 solutions to the linear system above, the vectors are linearly dependent. Choosing $c_3 = -1$ we obtain the third vector as a linear combination of the other two vectors,

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix} - \begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix}-1\\2\\5\end{bmatrix} = 2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix}.$$

◁

8.1.4. Exercises.

8.1.1.- .

8.1.2.- .

8.2. Matrix Algebra

The matrix-vector product introduced in Section 8.1 implies that an $n \times n$ matrix A is a function $A : \mathbb{R}^n \to \mathbb{R}^n$. This idea leads to introduce matrix operations, like the operations introduced for functions $f : \mathbb{R} \to \mathbb{R}$. These operations include linear combinations of matrices; the composition of matrices, also called matrix multiplication; the inverse of a square matrix; and the determinant of a square matrix. These operations will be needed in later Sections when we study systems of differential equations.

8.2.1. A matrix is a function.

The matrix-vector product leads to the interpretation that an $n \times n$ matrix A is a function. If we denote by $\mathbb{R}^n$ the space of all n-vectors, we see that the matrix-vector product associates to the n-vector $\boldsymbol{x}$ the unique n-vector $\boldsymbol{y} = A\boldsymbol{x}$. Therefore the matrix A determines a function $A : \mathbb{R}^n \to \mathbb{R}^n$.

Example 8.2.1: Describe the action on $\mathbb{R}^2$ of the function given by the 2×2 matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{8.2.1}$$

Solution: The action of this matrix on an arbitrary element $\boldsymbol{x} \in \mathbb{R}^2$ is given below,

$$A\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

Therefore, this matrix interchanges the components x_1 and x_2 of the vector $\boldsymbol{x}$. It can be seen in the first picture in Fig. 49 that this action can be interpreted as a reflection on the plane along the line $x_1 = x_2$. ◁

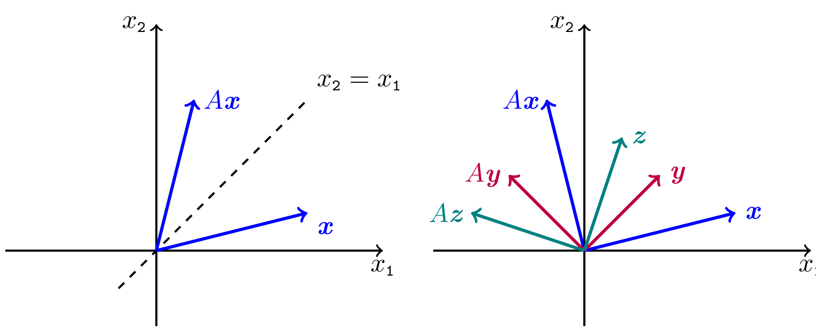


Figure 49. Geometrical meaning of the function determined by the matrix in Eq. (8.2.1) and the matrix in Eq. (8.2.2), respectively.

Example 8.2.2: Describe the action on $\mathbb{R}^2$ of the function given by the 2×2 matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.2.2}$$

Solution: The action of this matrix on an arbitrary element $\boldsymbol{x} \in \mathbb{R}^2$ is given below,

$$A\boldsymbol{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

In order to understand the action of this matrix, we give the following particular cases:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

These cases are plotted in the second figure on Fig. 49, and the vectors are called $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, respectively. We therefore conclude that this matrix produces a ninety degree counter-clockwise rotation of the plane. ◁

An example of a scalar-valued function is $f : \mathbb{R} \to \mathbb{R}$. We have seen here that an $n \times n$ matrix A is a function $A : \mathbb{R}^n \to \mathbb{R}^n$. Therefore, one can think an $n \times n$ matrix as a generalization of the concept of function from $\mathbb{R}$ to $\mathbb{R}^n$, for any positive integer n. It is well-known how to define several operations on scalar-valued functions, like linear combinations, compositions, and the inverse function. Therefore, it is reasonable to ask if these operation on scalar-valued functions can be generalized as well to matrices. The answer is yes, and the study of these and other operations is the subject of the rest of this Section.

8.2.2. **Matrix operations.** The linear combination of matrices refers to the addition of two matrices and the multiplication of a matrix by scalar. Linear combinations of matrices are defined component by component. For this reason we introduce the component notation for matrices and vectors. We denote an $m \times n$ matrix by $A = [A_{ij}]$, where A_{ij} are the components of matrix A, with $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Analogously, an n-vector is denoted by $\boldsymbol{v} = [v_j]$, where v_j are the components of the vector $\boldsymbol{v}$. We also introduce the notation $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$, that is, the set $\mathbb{F}$ can be the real numbers or the complex numbers.

Definition 8.2.1. *Let $A = [A_{ij}]$ and $B = [B_{ij}]$ be $m \times n$ matrices in $\mathbb{F}^{m,n}$ and a, b be numbers in $\mathbb{F}$. The* ***linear combination*** *of A and B is also and $m \times n$ matrix in $\mathbb{F}^{m,n}$, denoted as $a\,A + b\,B$, and given by*

$$a\,A + b\,B = [a\,A_{ij} + b\,B_{ij}].$$

The particular case where $a = b = 1$ corresponds to the addition of two matrices, and the particular case of $b = 0$ corresponds to the multiplication of a matrix by a number, that is,

$$A + B = [A_{ij} + B_{ij}], \qquad a\,A = [a\,A_{ij}].$$

Example 8.2.3: Find the $A + B$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$.

Solution: The addition of two equal size matrices is performed component-wise:

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} (1+2) & (2+3) \\ (3+5) & (4+1) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & 5 \end{bmatrix}.$$ ◁

Example 8.2.4: Find the $A + B$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: The matrices have different sizes, so their addition is not defined. ◁

Example 8.2.5: Find $2A$ and $A/3$, where $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

Solution: The multiplication of a matrix by a number is done component-wise, therefore

$$2A = 2 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}, \qquad \frac{A}{3} = \frac{1}{3} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{2}{3} & \frac{4}{3} & 2 \end{bmatrix}.$$ ◁

Since matrices are generalizations of scalar-valued functions, one can define operations on matrices that, unlike linear combinations, have no analogs on scalar-valued functions. One of such operations is the transpose of a matrix, which is a new matrix with the rows and columns interchanged.

Definition 8.2.2. *The* ***transpose*** *of a matrix* $A = [A_{ij}] \in \mathbb{F}^{m,n}$ *is the matrix denoted as* $A^T = \big[(A^T)_{kl}\big] \in \mathbb{F}^{n,m}$*, with its components given by* $\big(A^T\big)_{kl} = A_{lk}$*.*

EXAMPLE 8.2.6: Find the transpose of the 2×3 matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

SOLUTION: Matrix A has components A_{ij} with $i = 1, 2$ and $j = 1, 2, 3$. Therefore, its transpose has components $(A^T)_{ji} = A_{ij}$, that is, A^T has three rows and two columns,

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

◁

If a matrix has complex-valued coefficients, then the conjugate of a matrix can be defined as the conjugate of each component.

Definition 8.2.3. *The* ***complex conjugate*** *of a matrix* $A = [A_{ij}] \in \mathbb{F}^{m,n}$ *is the matrix* $\overline{A} = \big[\,\overline{A}_{ij}\big] \in \mathbb{F}^{m,n}$*.*

EXAMPLE 8.2.7: A matrix A and its conjugate is given below,

$$A = \begin{bmatrix} 1 & 2+i \\ -i & 3-4i \end{bmatrix}, \quad \Leftrightarrow \quad \overline{A} = \begin{bmatrix} 1 & 2-i \\ i & 3+4i \end{bmatrix}.$$

◁

EXAMPLE 8.2.8: A matrix A has real coefficients iff $A = \overline{A}$; It has purely imaginary coefficients iff $A = -\overline{A}$. Here are examples of these two situations:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \Rightarrow \quad \overline{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A;$$

$$A = \begin{bmatrix} i & 2i \\ 3i & 4i \end{bmatrix} \quad \Rightarrow \quad \overline{A} = \begin{bmatrix} -i & -2i \\ -3i & -4i \end{bmatrix} = -A.$$

◁

Definition 8.2.4. *The* ***adjoint*** *of a matrix* $A \in \mathbb{F}^{m,n}$ *is the matrix* $A^* = \overline{A}^{\,T} \in \mathbb{F}^{n,m}$*.*

Since $\big(\overline{A}\big)^T = \overline{(A^T)}$, the order of the operations does not change the result, that is why there is no parenthesis in the definition of A^*.

EXAMPLE 8.2.9: A matrix A and its adjoint is given below,

$$A = \begin{bmatrix} 1 & 2+i \\ -i & 3-4i \end{bmatrix}, \quad \Leftrightarrow \quad A^* = \begin{bmatrix} 1 & i \\ 2-i & 3+4i \end{bmatrix}.$$

◁

The transpose, conjugate and adjoint operations are useful to specify certain classes of matrices with particular symmetries. Here we introduce few of these classes.

Definition 8.2.5. *An* $n \times n$ *matrix* A *is called:*

(a) ***symmetric*** *iff holds* $A = A^T$*;*
(b) ***skew-symmetric*** *iff holds* $A = -A^T$*;*
(c) ***Hermitian*** *iff holds* $A = A^*$*;*
(d) ***skew-Hermitian*** *iff holds* $A = -A^*$*.*

Example 8.2.10: We present examples of each of the classes introduced in Def. 8.2.5.

Part (a): Matrices A and B are symmetric. Notice that A is also Hermitian, while B is not Hermitian,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 4 \\ 3 & 4 & 8 \end{bmatrix} = A^T, \qquad B = \begin{bmatrix} 1 & 2+3i & 3 \\ 2+3i & 7 & 4i \\ 3 & 4i & 8 \end{bmatrix} = B^T.$$

Part (b): Matrix C is skew-symmetric,

$$C = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} \quad \Rightarrow \quad C^T = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = -C.$$

Notice that the diagonal elements in a skew-symmetric matrix must vanish, since $C_{ij} = -C_{ji}$ in the case $i = j$ means $C_{ii} = -C_{ii}$, that is, $C_{ii} = 0$.

Part (c): Matrix D is Hermitian but is not symmetric:

$$D = \begin{bmatrix} 1 & 2+i & 3 \\ 2-i & 7 & 4+i \\ 3 & 4-i & 8 \end{bmatrix} \quad \Rightarrow \quad D^T = \begin{bmatrix} 1 & 2-i & 3 \\ 2+i & 7 & 4-i \\ 3 & 4+i & 8 \end{bmatrix} \neq D,$$

however,

$$D^* = \overline{D}^{\,T} = \begin{bmatrix} 1 & 2+i & 3 \\ 2-i & 7 & 4+i \\ 3 & 4-i & 8 \end{bmatrix} = D.$$

Notice that the diagonal elements in a Hermitian matrix must be real numbers, since the condition $A_{ij} = \overline{A}_{ji}$ in the case $i = j$ implies $A_{ii} = \overline{A}_{ii}$, that is, $2i\mathrm{Im}(A_{ii}) = A_{ii} - \overline{A}_{ii} = 0$. We can also verify what we said in part (a), matrix A is Hermitian since $A^* = \overline{A}^{\,T} = A^T = A$.

Part (d): The following matrix E is skew-Hermitian:

$$E = \begin{bmatrix} i & 2+i & -3 \\ -2+i & 7i & 4+i \\ 3 & -4+i & 8i \end{bmatrix} \quad \Rightarrow \quad E^T = \begin{bmatrix} i & -2+i & 3 \\ 2+i & 7i & -4+i \\ -3 & 4+i & 8i \end{bmatrix}$$

therefore,

$$E^* = \overline{E}^{\,T} \begin{bmatrix} -i & -2-i & 3 \\ 2-i & -7i & -4-i \\ -3 & 4-i & -8i \end{bmatrix} = -E.$$

A skew-Hermitian matrix has purely imaginary elements in its diagonal, and the off diagonal elements have skew-symmetric real parts with symmetric imaginary parts. ◁

The trace of a square matrix is a number, the sum of all the diagonal elements of the matrix.

Definition 8.2.6. *The **trace** of a square matrix $A = \big[A_{ij}\big] \in \mathbb{F}^{n,n}$, denoted as $\mathrm{tr}\,(A) \in \mathbb{F}$, is the sum of its diagonal elements, that is, the scalar given by $\mathrm{tr}\,(A) = A_{11} + \cdots + A_{nn}$.*

Example 8.2.11: Find the trace of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution: We only have to add up the diagonal elements:

$$\mathrm{tr}\,(A) = 1 + 5 + 9 \quad \Rightarrow \quad \mathrm{tr}\,(A) = 15.$$

◁

The operation of matrix multiplication originates in the composition of functions. We call it matrix multiplication instead of matrix composition because it reduces to the multiplication of real numbers in the case of 1×1 real matrices. Unlike the multiplication of real numbers, the product of general matrices is not commutative, that is, $AB \neq BA$ in the general case. This property reflects the fact that the composition of two functions is a non-commutative operation.

Definition 8.2.7. *The* ***matrix multiplication*** *of the $m \times n$ matrix $A = [A_{ij}]$ and the $n \times \ell$ matrix $B = [B_{jk}]$, where $i = 1, \cdots, m$, $j = 1, \cdots, n$ and $k = 1, \cdots, \ell$, is the $m \times \ell$ matrix AB given by*

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}. \tag{8.2.3}$$

The product is not defined for two arbitrary matrices, since the size of the matrices is important: The numbers of columns in the first matrix must match the numbers of rows in the second matrix.

$$\underset{m \times \boldsymbol{n}}{A} \quad \text{times} \quad \underset{\boldsymbol{n} \times \ell}{B} \quad \text{defines} \quad \underset{m \times \ell}{AB}$$

EXAMPLE 8.2.12: Compute AB, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

SOLUTION: The component $(AB)_{11} = 4$ is obtained from the first row in matrix A and the first column in matrix B as follows:

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{3} & 0 \\ \mathbf{2} & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & 1 \\ 1 & -2 \end{bmatrix}, \qquad (2)(3) + (-1)(2) = 4;$$

The component $(AB)_{12} = -1$ is obtained as follows:

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & \mathbf{0} \\ 2 & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} 4 & \mathbf{1} \\ 1 & -2 \end{bmatrix}, \qquad (2)(0) + (-1)(1) = -1;$$

The component $(AB)_{21} = 1$ is obtained as follows:

$$\begin{bmatrix} 2 & -1 \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{3} & 0 \\ \mathbf{2} & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ \mathbf{1} & -2 \end{bmatrix}, \qquad (-1)(3) + (2)(2) = 1;$$

And finally the component $(AB)_{22} = -2$ is obtained as follows:

$$\begin{bmatrix} 2 & -1 \\ \mathbf{-1} & \mathbf{2} \end{bmatrix} \begin{bmatrix} 3 & \mathbf{0} \\ 2 & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & \mathbf{-2} \end{bmatrix}, \qquad (-1)(0) + (2)(-1) = -2.$$

◁

EXAMPLE 8.2.13: Compute BA, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

SOLUTION: We find that $BA = \begin{bmatrix} 6 & -3 \\ 5 & -4 \end{bmatrix}$. Notice that in this case $AB \neq BA$. ◁

EXAMPLE 8.2.14: Compute AB and BA, where $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

SOLUTION: The product AB is

$$AB = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Rightarrow \quad AB = \begin{bmatrix} 16 & 23 & 30 \\ 6 & 9 & 12 \end{bmatrix}.$$

The product BA is not possible. ◁

8.2.3. **The inverse matrix.** We now introduce the concept of the inverse of a square matrix. Not every square matrix is invertible. The inverse of a matrix is useful to compute solutions to linear systems of algebraic equations.

Definition 8.2.8. *The matrix $I_n \in \mathbb{F}^{n,n}$ is the* ***identity matrix*** *iff $I_n \boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{F}^n$.*

It is simple to see that the components of the identity matrix are given by

$$I_n = [I_{ij}] \quad \text{with} \quad \begin{cases} I_{ii} = 1 \\ I_{ij} = 0 & i \neq j. \end{cases}$$

The cases $n = 2, 3$ are given by

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 8.2.9. *A matrix $A \in \mathbb{F}^{n,n}$ is called* ***invertible*** *iff there exists a matrix, denoted as A^{-1}, such that $(A^{-1})A = I_n$, and $A(A^{-1}) = I_n$.*

EXAMPLE 8.2.15: Verify that the matrix and its inverse are given by

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

SOLUTION: We have to compute the products,

$$A(A^{-1}) = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \Rightarrow \quad A(A^{-1}) = I_2.$$

It is simple to check that the equation $(A^{-1})A = I_2$ also holds. ◁

Theorem 8.2.10. *Given a 2×2 matrix A introduce the number Δ as follows,*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \Delta = ad - bc.$$

The matrix A is invertible iff $\Delta \neq 0$. Furthermore, if A is invertible, its inverse is given by

$$\mathsf{A}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{8.2.4}$$

The number Δ is called the determinant of A, since it is the number that determines whether A is invertible or not.

EXAMPLE 8.2.16: Compute the inverse of matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, given in Example 8.2.15.

SOLUTION: Following Theorem 8.2.10 we first compute $\Delta = 6 - 4 = 4$. Since $\Delta \neq 0$, then A^{-1} exists and it is given by

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

◁

EXAMPLE 8.2.17: Compute the inverse of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

SOLUTION: Following Theorem 8.2.10 we first compute $\Delta = 6 - 6 = 0$. Since $\Delta = 0$, then matrix A is not invertible. ◁

Gauss operations can be used to compute the inverse of a matrix. The reason for this is simple to understand in the case of 2×2 matrices, as can be seen in the following Example.

EXAMPLE 8.2.18: Given any 2×2 matrix A, find its inverse matrix, A^{-1}, or show that the inverse does not exist.

SOLUTION: If the inverse matrix, A^{-1} exists, then denote it as $A^{-1} = [\boldsymbol{x}_1,\ \boldsymbol{x}_2]$. The equation $A(A^{-1}) = I_2$ is then equivalent to $A\,[\boldsymbol{x}_1,\ \boldsymbol{x}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This equation is equivalent to solving two algebraic linear systems,

$$A\,\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad A\,\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Here is where we can use Gauss elimination operations. We use them on both systems

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \end{matrix} \end{array}\right], \qquad \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 1 \end{matrix} \end{array}\right].$$

However, we can solve both systems at the same time if we do Gauss operations on the bigger augmented matrix

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right].$$

Now, perform Gauss operations until we obtain the reduced echelon form for $[A|I_2]$. Then we can have two different types of results:

- If there is no line of the form $[0, 0|*, *]$ with any of the star coefficients non-zero, then matrix A is invertible and the solution vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ form the columns of the inverse matrix, that is, $A^{-1} = [\boldsymbol{x}_1, \boldsymbol{x}_2]$.
- If there is a line of the form $[0, 0|*, *]$ with any of the star coefficients non-zero, then matrix A is not invertible. ◁

EXAMPLE 8.2.19: Use Gauss operations to find the inverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

SOLUTION: As we said in the Example above, perform Gauss operation on the augmented matrix $[A|I_2]$ until the reduced echelon form is obtained, that is,

$$\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -4 & 1 & -2 \end{array}\right] \to$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right]$$

That is, matrix A is invertible and the inverse is

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad \Leftrightarrow \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

◁

EXAMPLE 8.2.20: Use Gauss operations to find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$.

Solution: We perform Gauss operations on the augmented matrix $[A|I_3]$ until we obtain its reduced echelon form, that is,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

We conclude that matrix A is invertible and

$$A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

◁

8.2.4. Determinants.

A determinant is a scalar computed form a square matrix that gives important information about the matrix, for example if the matrix is invertible or not. We now review the definition and properties of the determinant of 2×2 and 3×3 matrices.

Definition 8.2.11. *The* ***determinant of a*** 2×2 *matrix* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *is given by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The ***determinant of a*** 3×3 *matrix* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ *is given by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 8.2.21: The following three examples show that the determinant can be a negative, zero or positive number.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2, \qquad \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5, \qquad \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0.$$

The following is an example shows how to compute the determinant of a 3×3 matrix,

$$\begin{aligned} \begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} &= (1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ &= (1 - 2) - 3\,(2 - 3) - (4 - 3) \\ &= -1 + 3 - 1 \\ &= 1. \end{aligned}$$

◁

The absolute value of the determinant of a 2×2 matrix $A = [\boldsymbol{a}_1, \boldsymbol{a}_2]$ has a geometrical meaning: It is the area of the parallelogram whose sides are given by $\boldsymbol{a}_1$ and bia_2, that is, by the columns of the matrix A; see Fig. 50. Analogously, the absolute value of the determinant of a 3×3 matrix $A = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$ also has a geometrical meaning: It is the volume of the parallelepiped whose sides are given by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, that is, by the columns of the matrix A; see Fig. 50.

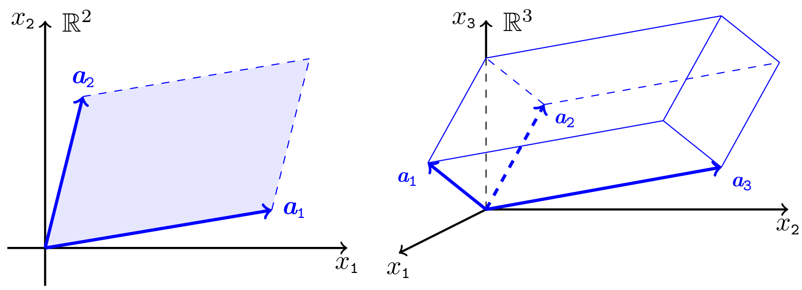


FIGURE 50. Geometrical meaning of the determinant.

The determinant of an $n \times n$ matrix A can be defined generalizing the properties that areas of parallelogram have in two dimensions and volumes of parallelepipeds have in three dimensions. One of these properties is the following: if one of the column vectors of the matrix A is a linear combination of the others, then the figure determined by these column vectors is not n-dimensional but $(n-1)$-dimensional, so its volume must vanish. We highlight this property of the determinant of $n \times n$ matrices in the following result.

Theorem 8.2.12. *The set of n-vectors $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n\}$, with $n \geqslant 1$, is linearly dependent iff*

$$\det[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n] = 0.$$

EXAMPLE 8.2.22: Show whether the set of vectors below linearly independent,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix} \right\}.$$

The determinant of the matrix whose column vectors are the vectors above is given by

$$\begin{vmatrix} 1 & 3 & -3 \\ 2 & 2 & 2 \\ 3 & 1 & 7 \end{vmatrix} = (1)\,(14-2) - 3\,(14-6) + (-3)\,(2-6) = 12 - 24 + 12 = 0.$$

Therefore, the set of vectors above is linearly dependent. ◁

The determinant of a square matrix also determines whether the matrix is invertible or not.

Theorem 8.2.13. *An $n \times n$ matrix A is invertible iff holds $\det(A) \neq 0$.*

EXAMPLE 8.2.23: Is matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$ invertible?

SOLUTION: We only need to compute the determinant of A.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{vmatrix} = (1) \begin{vmatrix} 5 & 7 \\ 7 & 9 \end{vmatrix} - (2) \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} + (3) \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}.$$

Using the definition of determinant for 2×2 matrices we obtain

$$\det(A) = (45 - 49) - 2(18 - 21) + 3(14 - 15) = -4 + 6 - 3.$$

Since $\det(A) = -1$, that is, non-zero, matrix A is invertible. $\lhd$

8.2.5. Exercises.

8.2.1.- .

8.2.2.- .

8.3. Diagonalizable Matrices

We continue with the review on Linear Algebra we started in Sections 8.1 and 8.2. We have seen that a square matrix is a function on the space of vectors. The matrix acts on a vector and the result is another vector. In this section we see that, given an $n \times n$ matrix, there may exist lines in $\mathbb{R}^n$ that are left invariant under the action of the matrix. The matrix acting on a non-zero vectors in such line is proportional to that vector. The vector is called an eigenvector of the matrix, and the proportionality factor is called an eigenvalue. If an $n \times n$ matrix has a linearly independent set containing n eigenvectors, then we call that matrix diagonalizable. In the next Section we will study linear differential systems with constant coefficients having a diagonalizable coefficients matrix. We will see that it is fairly simple to find solutions to such differential systems. The solutions can be expressed in terms of the exponential of the coefficient matrix. For that reason we study in this Section how to compute exponentials of square diagonalizable matrices.

8.3.1. Eigenvalues and eigenvectors.

When a square matrix acts on a vector the result is another vector that, more often than not, points in a different direction from the original vector. However, there may exist vectors whose direction is not changed by the matrix. We give these vectors a name.

Definition 8.3.1. *A non-zero n-vector $\boldsymbol{v}$ and a number λ are respectively called an **eigenvector** and **eigenvalue** of an $n \times n$ matrix A iff the following equation holds,*

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

We see that an eigenvector $\boldsymbol{v}$ determines a particular direction in the space $\mathbb{R}^n$, given by $(a\boldsymbol{v})$ for $a \in \mathbb{R}$, that remains invariant under the action of the function given by matrix A. That is, the result of matrix A acting on any vector $(a\boldsymbol{v})$ on the line determined by $\boldsymbol{v}$ is again a vector on the same line, as the following calculation shows it,

$$A(a\boldsymbol{v}) = a\,A\boldsymbol{v} = a\lambda\boldsymbol{v} = \lambda(a\boldsymbol{v}).$$

Example 8.3.1: Verify that the pair λ_1, $\boldsymbol{v}_1$ and the pair λ_2, $\boldsymbol{v}_2$ are eigenvalue and eigenvector pairs of matrix A given below,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \begin{cases} \lambda_1 = 4 & \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\[1ex] \lambda_2 = -2 & \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{cases}$$

Solution: We just must verify the definition of eigenvalue and eigenvector given above. We start with the first pair,

$$A\boldsymbol{v}_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \boldsymbol{v}_1 \quad \Rightarrow \quad A\boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1.$$

A similar calculation for the second pair implies,

$$A\boldsymbol{v}_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \boldsymbol{v}_2 \quad \Rightarrow \quad A\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2.$$

◁

Example 8.3.2: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: This is the matrix given in Example 8.2.1. The action of this matrix on the plane is a reflection along the line $x_1 = x_2$, as it was shown in Fig. 49. Therefore, this line

$x_1 = x_2$ is left invariant under the action of this matrix. This property suggests that an eigenvector is any vector on that line, for example

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 1.$$

So, we have found one eigenvalue-eigenvector pair: $\lambda_1 = 1$, with $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We remark that any non-zero vector proportional to $\boldsymbol{v}_1$ is also an eigenvector. Another choice fo eigenvalue-eigenvector pair is $\lambda_1 = 1$, with $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. It is not so easy to find a second eigenvector which does not belong to the line determined by $\boldsymbol{v}_1$. One way to find such eigenvector is noticing that the line perpendicular to the line $x_1 = x_2$ is also left invariant by matrix A. Therefore, any non-zero vector on that line must be an eigenvector. For example the vector $\boldsymbol{v}_2$ below, since

$$\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \lambda_2 = -1.$$

So, we have found a second eigenvalue-eigenvector pair: $\lambda_2 = -1$, with $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. These two eigenvectors are displayed on Fig. 51. ◁

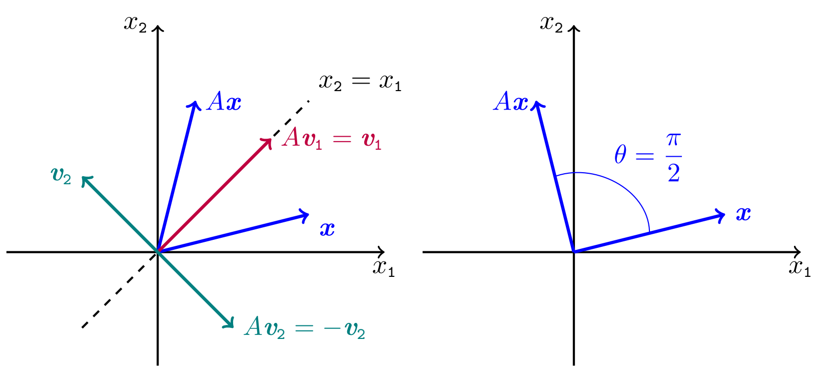


FIGURE 51. The first picture shows the eigenvalues and eigenvectors of the matrix in Example 8.3.2. The second picture shows that the matrix in Example 8.3.3 makes a counterclockwise rotation by an angle θ, which proves that this matrix does not have eigenvalues or eigenvectors.

There exist matrices that do not have eigenvalues and eigenvectors, as it is show in the example below.

EXAMPLE 8.3.3: Fix any number $\theta \in (0, 2\pi)$ and define the matrix $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Show that A has no real eigenvalues.

SOLUTION: One can compute the action of matrix A on several vectors and verify that the action of this matrix on the plane is a rotation counterclockwise by and angle θ, as shown in Fig. 51. A particular case of this matrix was shown in Example 8.2.2, where $\theta = \pi/2$. Since eigenvectors of a matrix determine directions which are left invariant by the action of the matrix, and a rotation does not have such directions, we conclude that the matrix A above does not have eigenvectors and so it does not have eigenvalues either. ◁

REMARK: We will show that matrix A in Example 8.3.3 has complex-valued eigenvalues.

We now describe a method to find eigenvalue-eigenvector pairs of a matrix, if they exit. In other words, we are going to solve the eigenvalue-eigenvector problem: Given an $n \times n$ matrix A find, if possible, all its eigenvalues and eigenvectors, that is, all pairs λ and $\boldsymbol{v} \neq \boldsymbol{0}$ solutions of the equation

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

This problem is more complicated than finding the solution $\boldsymbol{x}$ to a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where A and $\boldsymbol{b}$ are known. In the eigenvalue-eigenvector problem above neither λ nor $\boldsymbol{v}$ are known. To solve the eigenvalue-eigenvector problem for a matrix A we proceed as follows:

(a) First, find the eigenvalues λ;
(b) Second, for each eigenvalue λ find the corresponding eigenvectors $\boldsymbol{v}$.

The following result summarizes a way to solve the steps above.

Theorem 8.3.2 (Eigenvalues-eigenvectors).

(a) The number λ is an eigenvalue of an $n \times n$ matrix A iff holds,

$$\det(A - \lambda I) = 0. \tag{8.3.1}$$

(b) Given an eigenvalue λ of an $n \times n$ matrix A, the corresponding eigenvectors $\boldsymbol{v}$ are the non-zero solutions to the homogeneous linear system

$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}. \tag{8.3.2}$$

Proof of Theorem 8.3.2: The number λ and the non-zero vector $\boldsymbol{v}$ are an eigenvalue-eigenvector pair of matrix A iff holds

$$A\boldsymbol{v} = \lambda \boldsymbol{v} \quad \Leftrightarrow \quad (A - \lambda I)\boldsymbol{v} = \boldsymbol{0},$$

where I is the $n \times n$ identity matrix. Since $\boldsymbol{v} \neq \boldsymbol{0}$, the last equation above says that the columns of the matrix $(A - \lambda I)$ are linearly dependent. This last property is equivalent, by Theorem 8.2.12, to the equation

$$\det(A - \lambda I) = 0,$$

which is the equation that determines the eigenvalues λ. Once this equation is solved, substitute each solution λ back into the original eigenvalue-eigenvector equation

$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}.$$

Since λ is known, this is a linear homogeneous system for the eigenvector components. It always has non-zero solutions, since λ is precisely the number that makes the coefficient matrix $(A - \lambda I)$ not invertible. This establishes the Theorem. □

EXAMPLE 8.3.4: Find the eigenvalues λ and eigenvectors $\boldsymbol{v}$ of the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

SOLUTION: We first find the eigenvalues as the solutions of the Eq. (8.3.1). Compute

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{bmatrix}.$$

Then we compute its determinant,

$$0 = \det(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9 \quad \Rightarrow \quad \begin{cases} \lambda_{+} = 4, \\ \lambda_{-} = -2. \end{cases}$$

We have obtained two eigenvalues, so now we introduce $\lambda_{+} = 4$ into Eq. (8.3.2), that is,

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}.$$

Then we solve for $\boldsymbol{v}^{+}$ the equation

$$(A-4I)\boldsymbol{v}^{+}=\boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1^{+} \\ v_2^{+} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{+} = v_2^{+}, \\ v_2^{+} \quad \text{free.} \end{cases}$$

Al solutions to the equation above are then given by

$$\boldsymbol{v}^{+} = \begin{bmatrix} v_2^{+} \\ v_2^{+} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2^{+} \quad \Rightarrow \quad \boldsymbol{v}^{+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^{+}=1$. A similar calculation provides the eigenvector $\boldsymbol{v}^{-}$ associated with the eigenvalue $\lambda_{-}=-2$, that is, first compute the matrix

$$A+2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

then we solve for $\boldsymbol{v}^{-}$ the equation

$$(A+2I)\boldsymbol{v}^{-}=\boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1^{-} \\ v_2^{-} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution can be found using Gauss elimination operations, as follows,

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{-} = -v_2^{-}, \\ v_2^{-} \quad \text{free.} \end{cases}$$

All solutions to the equation above are then given by

$$\boldsymbol{v}^{-} = \begin{bmatrix} -v_2^{-} \\ v_2^{-} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2^{-} \quad \Rightarrow \quad \boldsymbol{v}^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

where we have chosen $v_2^{-}=1$. We therefore conclude that the eigenvalues and eigenvectors of the matrix A above are given by

$$\lambda_{+}=4, \quad \boldsymbol{v}^{+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_{-}=-2, \quad \boldsymbol{v}^{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

◁

It is useful to introduce few more concepts, that are common in the literature.

Definition 8.3.3. *The* ***characteristic polynomial*** *of an $n\times n$ matrix A is the function*

$$p(\lambda)=\det(A-\lambda I).$$

Example 8.3.5: Find the characteristic polynomial of matrix $A=\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: We need to compute the determinant

$$p(\lambda)=\det(A-\lambda I)=\begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2-9=\lambda^2-2\lambda+1-9.$$

We conclude that the characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda-8$. ◁

Since the matrix A in this example is 2×2, its characteristic polynomial has degree two. One can show that the characteristic polynomial of an $n\times n$ matrix has degree n. The eigenvalues of the matrix are the roots of the characteristic polynomial. Different matrices may have different types of roots, so we try to classify these roots in the following definition.

Definition 8.3.4. *Given an $n \times n$ matrix A with eigenvalues λ_i, with $i = 1, \cdots, k \leqslant n$, it is always possible to express the matrix characteristic polynomial as*

$$p(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_k)^{r_k}.$$

*The number r_i is called the **algebraic multiplicity** of the eigenvalue λ_i. Furthermore, the **geometric multiplicity** of the eigenvalue λ_i, denoted as s_i, is the maximum number of eigenvectors vectors corresponding to that eigenvalue λ_i forming a linearly independent set.*

Example 8.3.6: Find the eigenvalues algebraic and geometric multiplicities of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Solution: In order to find the algebraic multiplicity of the eigenvalues we need first to find the eigenvalues. We now that the characteristic polynomial of this matrix is given by

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 3 \\ 3 & (1-\lambda) \end{vmatrix} = (\lambda - 1)^2 - 9.$$

The roots of this polynomial are $\lambda_1 = 4$ and $\lambda_2 = -2$, so we know that $p(\lambda)$ can be rewritten in the following way,

$$p(\lambda) = (\lambda - 4)(\lambda + 2).$$

We conclude that the algebraic multiplicity of the eigenvalues are both one, that is,

$$\lambda_1 = 4, \quad r_1 = 1, \quad \text{and} \quad \lambda_2 = -2, \quad r_2 = 1.$$

In order to find the geometric multiplicities of matrix eigenvalues we need first to find the matrix eigenvectors. This part of the work was already done in the Example 8.3.4 above and the result is

$$\lambda_1 = 4, \quad \boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -2, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

From this expression we conclude that the geometric multiplicities for each eigenvalue are just one, that is,

$$\lambda_1 = 4, \quad s_1 = 1, \quad \text{and} \quad \lambda_2 = -2, \quad s_2 = 1.$$

◁

The following example shows that two matrices can have the same eigenvalues, and so the same algebraic multiplicities, but different eigenvectors with different geometric multiplicities.

Example 8.3.7: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: We start finding the eigenvalues, the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 0 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda - 1)(\lambda - 3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = -\dfrac{v_3^{(1)}}{2}, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free.} \end{cases}$$

Therefore, choosing $v_3^{(1)} = 2$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$, which are all solutions of the linear system

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} \quad \text{free,} \\ v_2^{(2)} \quad \text{free,} \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain two linearly independent solutions, the first one $\boldsymbol{v}^{(2)}$ with the choice $v_1^{(2)} = 1$, $v_2^{(2)} = 0$, and the second one $\boldsymbol{w}^{(2)}$ with the choice $v_1^{(2)} = 0$, $v_2^{(2)} = 1$, that is

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 2.$$

Summarizing, the matrix in this example has three linearly independent eigenvectors. ◁

EXAMPLE 8.3.8: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

SOLUTION: Notice that this matrix has only the coefficient a_{12} different from the previous example. Again, we start finding the eigenvalues, which are the roots of the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} (3-\lambda) & 1 & 1 \\ 0 & (3-\lambda) & 2 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = -(\lambda-1)(\lambda-3)^2 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 1, & r_1 = 1, \\ \lambda_2 = 3, & r_2 = 2. \end{cases}$$

So this matrix has the same eigenvalues and algebraic multiplicities as the matrix in the previous example. We now compute the eigenvector associated with the eigenvalue $\lambda_1 = 1$, which is the solution of the linear system

$$(A - I)\boldsymbol{v}^{(1)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \\ v_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(1)} = 0, \\ v_2^{(1)} = -v_3^{(1)}, \\ v_3^{(1)} \quad \text{free.} \end{cases}$$

Therefore, choosing $v_3^{(1)} = 1$ we obtain that

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \lambda_1 = 1, \qquad r_1 = 1, \qquad s_1 = 1.$$

In a similar way we now compute the eigenvectors for the eigenvalue $\lambda_2 = 3$. However, in this case we obtain only one solution, as this calculation shows,

$$(A - 3I)\boldsymbol{v}^{(2)} = \boldsymbol{0} \quad \Leftrightarrow \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1^{(2)} \\ v_2^{(2)} \\ v_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the few Gauss elimination operation we obtain the following,

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} v_1^{(2)} \quad \text{free,} \\ v_2^{(2)} = 0, \\ v_3^{(2)} = 0. \end{cases}$$

Therefore, we obtain only one linearly independent solution, which corresponds to the choice $v_1^{(2)} = 1$, that is,

$$\boldsymbol{v}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 3, \qquad r_2 = 2, \qquad s_2 = 1.$$

Summarizing, the matrix in this example has only two linearly independent eigenvectors, and in the case of the eigenvalue $\lambda_2 = 3$ we have the strict inequality

$$1 = s_2 < r_2 = 2.$$

$\lhd$

8.3.2. **Diagonalizable matrices.** We first introduce the notion of a diagonal matrix. Later on we define the idea of a diagonalizable matrix as a matrix that can be transformed into a diagonal matrix by a simple transformation.

Definition 8.3.5. *An $n \times n$ matrix A is called **diagonal** iff holds* $A = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}$.

That is, a matrix is diagonal iff every non-diagonal coefficient vanishes. From now on we use the following notation for a diagonal matrix A:

$$A = \operatorname{diag}\big[a_{11}, \cdots, a_{nn}\big] = \begin{bmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}.$$

This notation says that the matrix is diagonal and shows only the diagonal coefficients, since any other coefficient vanishes. Diagonal matrices are simple to manipulate since they share many properties with numbers. For example the product of two diagonal matrices is commutative. It is simple to compute power functions of a diagonal matrix. It is also simple to compute more involved functions of a diagonal matrix, like the exponential function.

EXAMPLE 8.3.9: For every positive integer n find A^n, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

SOLUTION: We start computing A^2 as follows,

$$A^2 = AA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}.$$

We now compute A^3,

$$A^3 = A^2 A = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix}.$$

Using induction, it is simple to see that $A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$. ◁

Many properties of diagonal matrices are shared by diagonalizable matrices. These are matrices that can be transformed into a diagonal matrix by a simple transformation.

Definition 8.3.6. *An $n \times n$ matrix A is called **diagonalizable** iff there exists an invertible matrix P and a diagonal matrix D such that*

$$A = PDP^{-1}.$$

Systems of linear differential equations are simple to solve in the case that the coefficient matrix is diagonalizable. In such case, it is simple to decouple the differential equations. One solves the decoupled equations, and then transforms back to the original unknowns.

EXAMPLE 8.3.10: We will see later on that matrix A is diagonalizable while B is not, where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad B = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}.$$

◁

There is a deep relation between the eigenvalues and eigenvectors of a matrix and the property of diagonalizability.

Theorem 8.3.7 (Diagonalizable matrices). *An $n \times n$ matrix A is diagonalizable iff matrix A has a linearly independent set of n eigenvectors. Furthermore,*

$$A = PDP^{-1}, \quad P = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n], \quad D = \operatorname{diag}\big[\lambda_1, \cdots, \lambda_n\big],$$

where $\lambda_i, \boldsymbol{v}_i$, for $i = 1, \cdots, n$, are eigenvalue-eigenvector pairs of matrix A.

Proof of Theorem 8.3.7:

($\Rightarrow$) Since matrix A is diagonalizable, there exist an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Multiply this equation by P^{-1} on the left and by P on the right, we get

$$D = P^{-1}AP. \tag{8.3.3}$$

Since $n \times n$ matrix D is diagonal, it has a linearly independent set of n eigenvectors, given by the column vectors of the identity matrix, that is,

$$D\boldsymbol{e}^{(i)} = d_{ii}\,\boldsymbol{e}^{(i)}, \qquad D = \operatorname{diag}\big[d_{11}, \cdots, d_{nn}\big], \qquad I = \big[\boldsymbol{e}^{(1)}, \cdots, \boldsymbol{e}^{(n)}\big].$$

So, the pair $d_{ii}, \boldsymbol{e}^{(i)}$ is an eigenvalue-eigenvector pair of D, for $i = 1 \cdots, n$. Using this information in Eq. (8.3.3) we get

$$d_{ii}\,\boldsymbol{e}^{(i)} = D\boldsymbol{e}^{(i)} = P^{-1}AP\boldsymbol{e}^{(i)} \quad \Rightarrow \quad A\big(P\boldsymbol{e}^{(i)}\big) = d_{ii}\,\big(P\boldsymbol{e}^{(i)}\big),$$

where the last equation comes from multiplying the former equation by P on the left. This last equation says that the vectors $\boldsymbol{v}^{(i)} = P\boldsymbol{e}^{(i)}$ are eigenvectors of A with eigenvalue d_{ii}. By definition, $\boldsymbol{v}^{(i)}$ is the i-th column of matrix P, that is,

$$P = \left[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\right].$$

Since matrix P is invertible, the eigenvectors set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. This establishes this part of the Theorem.

($\Leftarrow$) Let $\lambda_i, \boldsymbol{v}^{(i)}$ be eigenvalue-eigenvector pairs of matrix A, for $i = 1, \cdots, n$. Now use the eigenvectors to construct matrix $P = \left[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\right]$. This matrix is invertible, since the eigenvector set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. We now show that matrix $P^{-1}AP$ is diagonal. We start computing the product

$$AP = A\left[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\right] = \left[A\boldsymbol{v}^{(1)}, \cdots, A\boldsymbol{v}^{(n)}\right], = \left[\lambda_1\boldsymbol{v}^{(1)} \cdots, \lambda_n\boldsymbol{v}^{(n)}\right].$$

that is,

$$P^{-1}AP = P^{-1}\left[\lambda_1\boldsymbol{v}^{(1)}, \cdots, \lambda_n\boldsymbol{v}^{(n)}\right] = \left[\lambda_1 P^{-1}\boldsymbol{v}^{(1)}, \cdots, \lambda_n P^{-1}\boldsymbol{v}^{(n)}\right].$$

At this point it is useful to recall that P^{-1} is the inverse of P,

$$I = P^{-1}P \quad \Leftrightarrow \quad \left[\boldsymbol{e}^{(1)}, \cdots, \boldsymbol{e}^{(n)}\right] = P^{-1}\left[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\right] = \left[P^{-1}\boldsymbol{v}^{(1)}, \cdots, P^{-1}\boldsymbol{v}^{(n)}\right].$$

We conclude that $\boldsymbol{e}^{(i)} = P^{-1}\boldsymbol{v}^{(i)}$, for $i = 1 \cdots, n$. Using these equations in the equation for $P^{-1}AP$ we get

$$P^{-1}AP = \left[\lambda_1\boldsymbol{e}^{(1)}, \cdots, \lambda_n\boldsymbol{e}^{(n)}\right] = \operatorname{diag}\left[\lambda_1, \cdots, \lambda_n\right].$$

Denoting $D = \operatorname{diag}\left[\lambda_1, \cdots, \lambda_n\right]$ we conlude that $P^{-1}AP = D$, or equivalently

$$A = PDP^{-1}, \qquad P = \left[\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\right], \qquad D = \operatorname{diag}\left[\lambda_1, \cdots, \lambda_n\right].$$

This means that A is diagonalizable. This establishes the Theorem. □

Example 8.3.11: Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable.

Solution: We know that the eigenvalue eigenvector pairs are

$$\lambda_1 = 4, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce P and D as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

We must show that $A = PDP^{-1}$. This is indeed the case, since

$$PDP^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$PDP^{-1} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

We conclude, $PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \Rightarrow \quad PDP^{-1} = A$, that is, A is diagonalizable. ◁

EXAMPLE 8.3.12: Show that matrix $B = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$ is not diagonalizable.

SOLUTION: We first compute the matrix eigenvalues. The characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} \left(\frac{3}{2} - \lambda\right) & \frac{1}{2} \\ -\frac{1}{2} & \left(\frac{5}{2} - \lambda\right) \end{vmatrix} = \left(\frac{3}{2} - \lambda\right)\left(\frac{5}{2} - \lambda\right) + \frac{1}{4} = \lambda^2 - 4\lambda + 4.$$

The roots of the characteristic polynomial are computed in the usual way,

$$\lambda = \frac{1}{2}\left[4 \pm \sqrt{16 - 16}\right] \quad \Rightarrow \quad \lambda = 2, \qquad r = 2.$$

We have obtained a single eigenvalue with algebraic multiplicity $r = 2$. The associated eigenvectors are computed as the solutions to the equation $(A - 2I)\boldsymbol{v} = \boldsymbol{0}$. Then,

$$(A - 2I) = \begin{bmatrix} \left(\frac{3}{2} - 2\right) & \frac{1}{2} \\ -\frac{1}{2} & \left(\frac{5}{2} - 2\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad s = 1.$$

We conclude that the biggest linearly independent set of eigenvalues for the 2×2 matrix B contains only one vector, insted of two. Therefore, matrix B is not diagonalizable. ◁

Because of Theorem 8.3.7 it is important in applications to know whether an $n \times n$ matrix has a linearly independent set of n eigenvectors. More often than not there is no simple way to check this property other than to compute all the matrix eigenvectors. However, there is a simple particular case: When the $n \times n$ matrix has n different eigenvalues. In such case we do not need to compute the eigenvectors. The following result says that such matrix always have a linearly independent set of n eigenvectors, and so, by Theorem 8.3.7, will be diagonalizable.

Theorem 8.3.8 (Different eigenvalues). *If an $n \times n$ matrix has n different eigenvalues, then this matrix has a linearly independent set of n eigenvectors.*

Proof of Theorem 8.3.8: Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A, all different from each other. Let $\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}$ the corresponding eigenvectors, that is, $A\boldsymbol{v}^{(i)} = \lambda_i \boldsymbol{v}^{(i)}$, with $i = 1, \cdots, n$. We have to show that the set $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ is linearly independent. We assume that the opposite is true and we obtain a contradiction. Let us assume that the set above is linearly dependent, that is, there exists constants $c_1, \cdots, c_n$, not all zero, such that,

$$c_1 \boldsymbol{v}^{(1)} + \cdots + c_n \boldsymbol{v}^{(n)} = \boldsymbol{0}. \tag{8.3.4}$$

Let us name the eigenvalues and eigenvectors such that $c_1 \neq 0$. Now, multiply the equation above by the matrix A, the result is,

$$c_1 \lambda_1 \boldsymbol{v}^{(1)} + \cdots + c_n \lambda_n \boldsymbol{v}^{(n)} = \boldsymbol{0}.$$

Multiply Eq. (8.3.4) by the eigenvalue λ_n, the result is,

$$c_1 \lambda_n \boldsymbol{v}^{(1)} + \cdots + c_n \lambda_n \boldsymbol{v}^{(n)} = \boldsymbol{0}.$$

Subtract the second from the first of the equations above, then the last term on the right-hand sides cancels out, and we obtain,

$$c_1 (\lambda_1 - \lambda_n) \boldsymbol{v}^{(1)} + \cdots + c_{n-1} (\lambda_{n-1} - \lambda_n) \boldsymbol{v}^{(n-1)} = \boldsymbol{0}. \tag{8.3.5}$$

Repeat the whole procedure starting with Eq. (8.3.5), that is, multiply this later equation by matrix A and also by λ_{n-1}, then subtract the second from the first, the result is,

$$c_1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1})\boldsymbol{v}^{(1)} + \cdots + c_{n-2}(\lambda_{n-2} - \lambda_n)(\lambda_{n-2} - \lambda_{n-1})\boldsymbol{v}^{(n-2)} = \boldsymbol{0}.$$

Repeat the whole procedure a total of $n-1$ times, in the last step we obtain the equation

$$c_1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1}) \cdots (\lambda_1 - \lambda_2)\boldsymbol{v}^{(1)} = \boldsymbol{0}.$$

Since all the eigenvalues are different, we conclude that $c_1 = 0$, however this contradicts our assumption that $c_1 \neq 0$. Therefore, the set of n eigenvectors must be linearly independent. □

Example 8.3.13: Is matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ diagonalizable?

Solution: We compute the matrix eigenvalues, starting with the characteristic polynomial,

$$p(\lambda) = \begin{vmatrix} (1-\lambda) & 1 \\ 1 & (1-\lambda) \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda \quad \Rightarrow \quad p(\lambda) = \lambda(\lambda - 2).$$

The roots of the characteristic polynomial are the matrix eigenvalues,

$$\lambda_1 = 0, \qquad \lambda_2 = 2.$$

The eigenvalues are different, so by Theorem 8.3.8, matrix A is diagonalizable. ◁

8.3.3. **The exponential of a matrix.** Functions of a diagonalizable matrix are simple to compute in the case that the function admits a Taylor series expansion. One of such functions is the exponential function. In this Section we compute the exponential of a diagonalizable matrix. However, we emphasize that this method can be trivially extended to any function admitting a Taylor series expansion. A key result needed to compute the Taylor series expansion of a matrix is the n-power of the matrix.

Theorem 8.3.9. *If the $n \times n$ matrix A is diagonalizable, with invertible matrix P and diagonal matrix D satisfying $A = PDP^{-1}$, then for every integer $n \geqslant 1$ holds*

$$A^n = PD^nP^{-1}. \tag{8.3.6}$$

Proof of Theorem 8.3.9: It is not difficult to generalize the calculation done in Example 8.3.9 to obtain the n-th power of a diagonal matrix $D = \operatorname{diag}\big[d_1, \cdots, d_n\big]$, and the result is another diagonal matrix given by

$$D^n = \operatorname{diag}\big[d_1^n, \cdots, d_n^n\big].$$

We use this result and induction in n to prove Eq.(8.3.6). Since the case $n = 1$ is trivially true, we start computing the case $n = 2$. We get

$$A^2 = \big(PDP^{-1}\big)^2 = \big(PDP^{-1}\big)\big(PDP^{-1}\big) = PDDP^{-1} \quad \Rightarrow \quad A^2 = PD^2P^{-1},$$

that is, Eq. (8.3.6) holds for $n = 2$. Now, assuming that this Equation holds for n, we show that it also holds for $n+1$. Indeed,

$$A^{(n+1)} = A^nA = \big(PD^nP^{-1}\big)\big(PDP^{-1}\big) = PD^nP^{-1}PDP^{-1} = PD^nDP^{-1}.$$

We conclude that $A^{(n+1)} = PD^{(n+1)}P^{-1}$. This establishes the Theorem. □

The exponential function $f(x) = e^x$ is usually defined as the inverse of the natural logarithm function $g(x) = \ln(x)$, which in turns is defined as the area under the graph of the function $h(x) = 1/x$ from 1 to x, that is,

$$\ln(x) = \int_1^x \frac{1}{y}\, dy, \qquad x \in (0, \infty).$$

It is not clear how to extend to matrices this way of defining the exponential function on real numbers. However, the exponential function on real numbers satisfies several identities that can be used as definition for the exponential on matrices. One of these identities is the Taylor series expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

This identity is the key to generalize the exponential function to diagonalizable matrices.

Definition 8.3.10. *The* ***exponential*** *of a square matrix A is defined as the infinite sum*

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{8.3.7}$$

One must show that the definition makes sense, that is, that the infinite sum in Eq. (8.3.7) converges. We show in these notes that this is the case when matrix A is diagonal and when matrix A is diagonalizable. The case of non-diagonalizable matrix is more difficult to prove, and we do not do it in these notes.

Theorem 8.3.11. *If $D = \text{diag}\big[d_1, \cdots, d_n\big]$, then holds $e^D = \text{diag}\big[e^{d_1}, \cdots, e^{d_n}\big]$.*

Proof of Theorem 8.3.11: It is simple to see that the infinite sum in Eq (8.3.7) converges for diagonal matrices. Start computing

$$e^D = \sum_{k=0}^{\infty} \frac{1}{k!} \big(\text{diag}\big[d_1, \cdots, d_n\big]\big)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \text{diag}\big[(d_1)^k, \cdots, (d_n)^k\big] .$$

Since each matrix in the sum on the far right above is diagonal, then holds

$$e^D = \text{diag}\Big[\sum_{k=0}^{\infty} \frac{(d_1)^k}{k!}, \cdots, \sum_{k=0}^{\infty} \frac{(d_n)^k}{k!}\Big].$$

Now, each sum in the diagonal of matrix above satisfies $\displaystyle\sum_{k=0}^{\infty} \frac{(d_i)^k}{k!} = e^{d_i}$. Therefore, we arrive to the equation $e^D = \text{diag}\big[e^{d_1}, \cdots, e^{d_n}\big]$. This establishes the Theorem. □

EXAMPLE 8.3.14: Compute e^A, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

SOLUTION: Theorem 8.3.11 implies that $e^A = \begin{bmatrix} e^2 & 0 \\ 0 & e^7 \end{bmatrix}$. ◁

The case of diagonalizable matrices is more involved and is summarized below.

Theorem 8.3.12. *If the $n \times n$ matrix A is diagonalizable, with invertible matrix P and diagonal matrix D satisfying $A = PDP^{-1}$, then the exponential of matrix A is given by*

$$e^A = Pe^D P^{-1}. \tag{8.3.8}$$

The formula above says that to find the exponential of a diagonalizable matrix there is no need to compute the infinite sum in Definition 8.3.10. To compute the exponential of a diagonalizable matrix it is only needed to compute the product of three matrices. It also says that to compute the exponential of a diagonalizable matrix we need to compute the eigenvalues and eigenvectors of the matrix.

Proof of Theorem 8.3.12: We again start with Eq. (8.3.7),

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^n = \sum_{k=0}^{\infty} \frac{1}{k!} (PDP^{-1})^n = \sum_{k=0}^{\infty} \frac{1}{k!} (PD^nP^{-1}),$$

where the last step comes from Theorem 8.3.9. Now, in the expression on the far right we can take common factor P on the left and P^{-1} on the right, that is,

$$e^A = P\Big(\sum_{k=0}^{\infty} \frac{1}{k!} D^n\Big) P^{-1}.$$

The terms between brackets sum up to the exponential of the diagonal matrix D, that is,

$$e^A = Pe^DP^{-1}.$$

Since the exponential of a diagonal matrix is computed in Theorem 8.3.11, this establishes the Theorem. □

Example 8.3.15: Compute e^A, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

Solution: To compute the exponential of matrix A above we need the decomposition $A = PDP^{-1}$. Matrices P and D are constructed with the eigenvectors and eigenvalues of matrix A, respectively. From Example 8.3.4 we know that

$$\lambda_1 = 4, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -2, \quad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Introduce P and D as follows,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then, the exponential function is given by

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Usually one leaves the function in this form. If we multiply the three matrices out we get

$$e^{At} = \frac{1}{2} \begin{bmatrix} (e^{4t} + e^{-2t}) & (e^{4t} - e^{-2t}) \\ (e^{4t} - e^{-2t}) & (e^{4t} + e^{-2t}) \end{bmatrix}.$$

◁

The exponential of an $n \times n$ matrix A can be used to define the matrix-valued function with values e^{At}. In the case that the matrix A is diagonalizable we obtain

$$e^{At} = Pe^{Dt}P^{-1},$$

where $e^{Dt} = \text{diag}\big[e^{d_1 t}, \cdots, e^{d_n t}\big]$ and $D = \text{diag}\big[d_1, \cdots, d_n\big]$. It is not difficult to show that

$$\frac{de^{At}}{dt} = A\,e^{At} = e^{At}\,A.$$

This exponential function will be useful to express solutions of a linear homogeneous differential system $\boldsymbol{x}' = A\boldsymbol{x}$. This is the subject of the next Section.

8.3.4. Exercises.

8.3.1.- .

8.3.2.- .

Chapter 9. Appendices

Appendix A. Review Complex Numbers

Coming up.

Appendix B. Review of Power Series

We summarize a few results on power series that we will need to find solutions to differential equations. A more detailed presentation of these ideas can be found in standard calculus textbooks, [1, 2, 11, 13]. We start with the definition of analytic functions, which are functions that can be written as a power series expansion on an appropriate domain.

Definition B.1. *A function y is **analytic** on an interval $(x_0-\rho, x_0+\rho)$ iff it can be written as the power series expansion below, convergent for $|x-x_0|<\rho$,*

$$y(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^n.$$

Example B.1: We show a few examples of analytic functions on appropriate domains.

(a) The function $y(x) = \dfrac{1}{1-x}$ is analytic on the interval $(-1,1)$, because it has the power series expansion centered at $x_0 = 0$, convergent for $|x|<1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots.$$

It is clear that this series diverges for $x \geqslant 1$, but it is not obvious that this series converges if and only if $|x|<1$.

(b) The function $y(x) = e^x$ is analytic on $\mathbb{R}$, and can be written as the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

(c) A function y having at x_0 both infinitely many continuous derivatives and a convergent power series is analytic where the series converges. The Taylor expansion centered at x_0 of such a function is

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}\,(x-x_0)^n,$$

and this means

$$y(x) = y(x_0) + y'(x_0)\,(x-x_0) + \frac{y''(x_0)}{2!}\,(x-x_0)^2 + \frac{y'''(x_0)}{3!}\,(x-x_0)^3 + \cdots.$$

◁

The Taylor series can be very useful to find the power series expansions of function having infinitely many continuous derivatives.

Example B.2: Find the Taylor series of $y(x) = \sin(x)$ centered at $x_0 = 0$.

Solution: We need to compute the derivatives of the function y and evaluate these derivatives at the point we center the expansion, in this case $x_0 = 0$.

$$y(x) = \sin(x) \quad\Rightarrow\quad y(0) = 0, \qquad y'(x) = \cos(x) \quad\Rightarrow\quad y'(0) = 1,$$
$$y''(x) = -\sin(x) \quad\Rightarrow\quad y''(0) = 0, \qquad y'''(x) = -\cos(x) \quad\Rightarrow\quad y'''(0) = -1.$$

One more derivative gives $y^{(4)}(t) = \sin(t)$, so $y^{(4)} = y$, the cycle repeats itself. It is not difficult to see that the Taylor's formula implies,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad\Rightarrow\quad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,x^{(2n+1)}.$$

◁

Remark: The Taylor series at $x_0 = 0$ for $y(x) = \cos(x)$ is computed in a similar way,

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{(2n)}.$$

Elementary functions like quotient of polynomials, trigonometric functions, exponential and logarithms can be written as power series. But the power series of any of these functions may not be defined on the whole domain of the function. The following example shows a function with this property.

Example B.3: Find the Taylor series for $y(x) = \dfrac{1}{1-x}$ centered at $x_0 = 0$.

Solution: Notice that this function is well defined for every $x \in \mathbb{R} - \{1\}$. The function graph can be seen in Fig. **??**. To find the Taylor series we need to compute the n-derivative, $y^{(n)}(0)$. It simple to check that,

$$y^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad \text{so} \quad y^{(n)}(0) = n!.$$

We conclude that: $y(x) = \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$.

One can prove that this power series converges if and only if $|x| < 1$. ◁

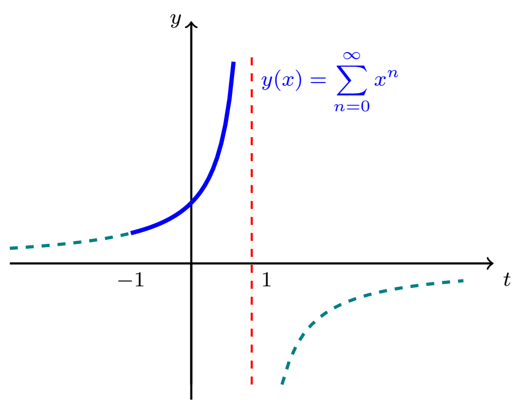


Figure 52. The graph of $y = \dfrac{1}{(1-x)}$.

Remark: The power series $y(x) = \displaystyle\sum_{n=0}^{\infty} x^n$ does not converge on $(-\infty, -1] \cup [1, \infty)$. But there are different power series that converge to $y(x) = \dfrac{1}{1-x}$ on intervals inside that domain. For example the Taylor series about $x_0 = 2$ converges for $|x-2| < 1$, that is $1 < x < 3$.

$$y^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \quad \Rightarrow \quad y^{(n)}(2) = \frac{n!}{(-1)^{n+1}} \quad \Rightarrow \quad y(x) = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n.$$

Later on we might need the notion of convergence of an infinite series in absolute value.

Definition B.2. *The power series $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n\,(x-x_0)^n$ **converges in absolute value** iff the series $\displaystyle\sum_{n=0}^{\infty} |a_n|\,|x-x_0|^n$ converges.*

Remark: If a series converges in absolute value, it converges. The converse is not true.

EXAMPLE B.4: One can show that the series $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but this series does not converge absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. See [11, 13]. ◁

Since power series expansions of functions might not converge on the same domain where the function is defined, it is useful to introduce the region where the power series converges.

Definition B.3. *The* ***radius of convergence*** *of a power series* $y(x) = \sum_{n=0}^{\infty} a_n \,(x - x_0)^n$ *is the number* $\rho \geqslant 0$ *satisfying both the series converges absolutely for* $|x - x_0| < \rho$ *and the series diverges for* $|x - x_0| > \rho$.

Remark: The radius of convergence defines the size of the biggest open interval where the power series converges. This interval is symmetric around the series center point x_0.

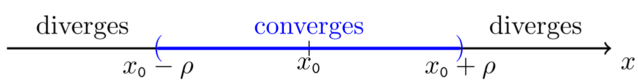


FIGURE 53. Example of the radius of convergence.

EXAMPLE B.5: We state the radius of convergence of few power series. See [11, 13].

(1) The series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ has radius of convergence $\rho = 1$.

(2) The series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence $\rho = \infty$.

(3) The series $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{(2n+1)}$ has radius of convergence $\rho = \infty$.

(4) The series $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{(2n)}$ has radius of convergence $\rho = \infty$.

(5) The series $\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{(2n+1)}$ has radius of convergence $\rho = \infty$.

(6) The series $\cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{(2n)}$ has radius of convergence $\rho = \infty$.

One of the most used tests for the convergence of a power series is the ratio test.

Theorem B.4 (Ratio Test). *Given the power series* $y(x) = \sum_{n=0}^{\infty} a_n \,(x - x_0)^n$, *introduce the number* $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. *Then, the following statements hold:*

(1) The power series converges in the domain $|x - x_0| L < 1$.

(2) The power series diverges in the domain $|x - x_0|L > 1$.

(3) The power series may or may not converge at $|x - x_0|L = 1$.

Therefore, if $L \neq 0$, then $\rho = \dfrac{1}{L}$ is the series radius of convergence; if $L = 0$, then the radius of convergence is $\rho = \infty$.

Remark: The convergence of the power series at $x_0 + \rho$ and $x_0 - \rho$ needs to be studied on each particular case.

Power series are usually written using summation notation. We end this review mentioning a few summation index manipulations, which are fairly common. Take the series

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots ,$$

which is usually written using the summation notation

$$y(x) = \sum_{n=0}^{\infty} a_n \,(x - x_0)^n.$$

The label name, n, has nothing particular, any other label defines the same series. For example the labels k and m below,

$$y(x) = \sum_{k=0}^{\infty} a_k \,(x - x_0)^k = \sum_{m=-3}^{\infty} a_{m+3} \,(x - x_0)^{m+3}.$$

In the first sum we just changed the label name from n to k, that is, $k = n$. In the second sum above we relabel the sum, $n = m + 3$. Since the initial value for n is $n = 0$, then the initial value of m is $m = -3$. Derivatives of power series can be computed derivating every term in the power series,

$$y'(x) = \sum_{n=0}^{\infty} n\, a_n \,(x - x_0)^{n-1} = \sum_{n=1}^{\infty} n\, a_n \,(x - x_0)^{n-1} = a_1 + 2a_2(x - x_0) + \cdots .$$

The power series for the y' can start either at $n = 0$ or $n = 1$, since the coefficients have a multiplicative factor n. We will usually relabel derivatives of power series as follows,

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n \,(x - x_0)^{n-1} = \sum_{m=0}^{\infty} (m + 1)\, a_{m+1} \,(x - x_0)^m$$

where $m = n - 1$, that is, $n = m + 1$.

Appendix C. Review Exercises

Coming up.

Appendix D. Practice Exams

Coming up.

Appendix E. Answers to Exercises

Chapter 1: First order equations

Section 1.1: Linear constant coefficients equations

1.1.1.- It is simple to see that:

$$y' = 2(t+2)\,e^{2t} + e^{2t},$$

and that

$$2y + e^{2t} = 2(t+2)\,e^{2t} + e^{2t}.$$

Hence $y' = 2y + e^{2t}$. Also holds that $y(0) = (0+2)\,e^0 = 2$.

1.1.2.- $y(t) = c\,e^{-4t} + \dfrac{1}{2}$, with $c \in R$.

1.1.3.- $y(t) = \dfrac{9}{2}\,e^{-4t} + \dfrac{1}{2}$.

1.1.4.- $y(x) = c\,e^{6t} - \dfrac{1}{6}$, with $c \in R$.

1.1.5.- $y(x) = \dfrac{7}{6}\,e^{6t} - \dfrac{1}{6}$.

1.1.6.- $y(t) = \dfrac{1}{3}\,e^{3(t-1)} + \dfrac{2}{3}$.

Section 1.2: Linear variable coefficients equations

1.2.1.- $y(t) = 4e^{-t} - e^{-2t}$.

1.2.2.- $y(t) = 2e^{t} + 2(t-1)\,e^{2t}$.

1.2.3.- $y(t) = \dfrac{\pi}{2t^2} - \dfrac{\cos(t)}{t^2}$.

1.2.4.- $y(t) = c\,e^{(1+t^2)^2}$, with $c \in \mathbb{R}$.

1.2.5.- $y(t) = \dfrac{t^2}{n+2} + \dfrac{c}{t^n}$, with $c \in \mathbb{R}$.

1.2.6.- $y(t) = c\,e^{t^2}$, with $c \in \mathbb{R}$.
Let y_1 and y_2 be solutions, that is, $2ty_1 - y_1' = 0$ and $2ty_2 - y_2' = 0$. Then

$$2t(y_1 + y_2) - (y_1 + y_2)' =$$

$$(2ty_1 - y_1') + (2ty_2 - y_2') = 0.$$

1.2.7.- Define $v(t) = 1/y(t)$. The equation for v is $v' = t\,v - t$. Its solution is $v(t) = c\,e^{t^2/2} + 1$. Therefore,

$$y(t) = \frac{1}{t\,e^{t^2/2} + 1}.$$

1.2.8.- $y(x) = \big(6 + c\,e^{-x^2/4}\big)^2$

Section 1.3: Separable equations

1.3.1.- Implicit form: $\dfrac{y^2}{2} = \dfrac{t^3}{3} + c$.

Explicit form: $y = \pm\sqrt{\dfrac{2t^3}{3} + 2c}$.

1.3.2.- $y^4 + y = t - t^3 + c$, with $c \in \mathbb{R}$.

1.3.3.- $y(t) = \dfrac{3}{3 - t^3}$.

1.3.4.- $y(t) = c\,e^{\sqrt{1+t^2}}$.

1.3.5.- $y(t) = t\big(\ln(|t|) + c\big)$.

1.3.6.- $y^2(t) = 2t^2\big(\ln(|t|) + c\big)$.

1.3.7.- Implicit: $y^2 + ty - 2t = 0$.
Explicit: $y(t) = \dfrac{1}{2}\big(-t + \sqrt{t^2 + 8t}\big)$.

1.3.8.- Hint: Recall the Definition 1.3.4 and use that

$$y_1'(x) = f\big(x, y_1(x)\big),$$

for any independent variable x, for example for $x = kt$.

Section 1.4: Exact equations

1.4.1.-

(a) The equation is exact. $N = (1+t^2)$, $M = 2t\,y$, so $\partial_t N = 2t = \partial_y M$.

(b) Since a potential function is given by $\psi(t,y) = t^2\,y + y$, the solution is

$$y(t) = \frac{c}{t^2+1}, \quad c \in \mathbb{R}.$$

1.4.2.-

(a) The equation is exact. We have $N = t\cos(y) - 2y$, $M = t + \sin(y)$,

$$\partial_t N = \cos(y) = \partial_y M.$$

(b) Since a potential function is given by $\psi(t,y) = \dfrac{t^2}{2} + t\sin(y) - y^2$, the solution is

$$\frac{t^2}{2} + t\sin(y(t)) - y^2(t) = c,$$

for $c \in \mathbb{R}$.

1.4.3.-

(a) The equation is exact. We have $N = -2y + t\,e^{ty}$, $M = 2 + y\,e^{ty}$,

$$\partial_t N = (1 + t\,y)\,e^{ty} = \partial_y M.$$

(b) Since the potential function is given by $\psi(t,y) = 2t + e^{ty} - y^2$, the solution is

$$2t + e^{t\,y(t)} - y^2(t) = c,$$

for $c \in \mathbb{R}$.

1.4.4.-

(a) $\mu(x) = 1/x$.

(b) $y^3 - 3xy + \dfrac{18}{5}\,x^5 = 1$.

1.4.5.-

(a) $\mu(x) = x^2$.

(b) $y = -\dfrac{2}{\sqrt{1+2x^4}}$. The negative square root is selected because the the initial condition is $y(0) < 0$.

Section 1.5: Applications

1.5.1.-

(a) Denote $m(t)$ the material mass as function of time. Use m in mgr and t in hours. Then

$$m(t) = m_0\, e^{-kt},$$

where $m_0 = 50$ mgr and $k = \ln(5)$ hours.

(b) $m(4) = \dfrac{2}{25}$ mgr.

(c) $\tau = \dfrac{\ln(2)}{\ln(5)}$ hours, so $\tau \simeq 0.43$ hours.

1.5.2.- Since

$$Q(t) = Q_0 e^{-(r_o/V_0)t},$$

the condition

$$Q_1 = Q_0 e^{-(r_o/V_0)t_1}$$

implies that

$$t_1 = \frac{V_0}{r_o} \ln\Big(\frac{Q_0}{Q_1}\Big).$$

Therefore, $t = 20 \ln(5)$ minutes.

1.5.3.- Since

$$Q(t) = V_0 q_i \big(1 - e^{-(r_o/V_0)t}\big)$$

and

$$\lim_{t\to\infty} Q(t) = V_0 q_i,$$

the result in this problem is

$$Q(t) = 300\big(1 - e^{-t/50}\big)$$

and

$$\lim_{t\to\infty} Q(t) = 300 \text{ grams}.$$

1.5.4.- Denoting $\Delta r = r_i - r_o$ and $V(t) = \Delta r\, t + V_0$, we obtain

$$Q(t) = \Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}} Q_0 + q_i \Big[V(t) - V_0 \Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}}\Big].$$

A reordering of terms gives

$$Q(t) = q_i V(t) - \Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}} (q_i V_0 - Q_0)$$

and replacing the problem values yields

$$Q(t) = t + 200 - 100\,\frac{(200)^2}{(t+200)^2}.$$

The concentration $q(t) = Q(t)/V(t)$ is

$$q(t) = q_i - \Big[\frac{V_0}{V(t)}\Big]^{\frac{r_o}{\Delta r}+1} \Big(q_i - \frac{Q_0}{V_0}\Big).$$

The concentration at $V(t) = V_m$ is

$$q_m = q_i - \Big[\frac{V_0}{V_m}\Big]^{\frac{r_o}{\Delta r}+1} \Big(q_i - \frac{Q_0}{V_0}\Big),$$

which gives the value

$$q_m = \frac{121}{125} \text{ grams/liter}.$$

In the case of an unlimited capacity, $\lim_{t\to\infty} V(t) = \infty$, thus the equation for $q(t)$ above says

$$\lim_{t\to\infty} q(t) = q_i.$$

Section 1.6: Non-linear equations

1.6.1.-

(a) Write the equation as

$$y' = -\frac{2\ln(t)}{(t^2-4)}\, y.$$

The equation is not defined for

$$t = 0 \quad t = \pm 2.$$

This provides the intervals

$$(-\infty, -2),\ (-2, 2),\ (2, \infty).$$

Since the initial condition is at $t = 1$, the interval where the solution is defined is

$$D = (0, 2).$$

(b) The equation is not defined for

$$t = 0, \quad t = 3.$$

This provides the intervals

$$(-\infty, 0),\ (0, 3),\ (3, \infty).$$

Since the initial condition is at $t = -1$, the interval where the solution is defined is

$$D = (-\infty, 0).$$

1.6.2.-

(a) $y = \dfrac{2}{3}t$.

(b) Outside the disk $t^2 + y^2 \leqslant 1$.

1.6.3.-

(a) Since $y = \sqrt{y_0^2 - 4t^2}$, and the initial condition is at $t = 0$, the solution domain is

$$D = \Big[-\frac{y_0}{2}, \frac{y_0}{2}\Big].$$

(b) Since $y = \dfrac{y_0}{1 - t^2 y_0}$ and the initial condition is at $t = 0$, the solution domain is

$$D = \Big[-\frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{y_0}}\Big].$$

Chapter 2: Second order linear equations

Section 2.1: Variable coefficients

2.1.1.- .

2.1.2.- .

Section ??: Constant coefficients

??.1.- .

??.2.- .

Section ??: Complex roots

??.1.- .

??.2.- .

Section ??: Repeated roots

??.??.- .

??.??.- .

Section ??: Undetermined coefficients

??.??.- .

??.??.- .

Section ??: Variation of parameters

??.??.- .

??.??.- .

Chapter 3: Power series solutions

Section 3.1: Regular points

3.1.1.- .

3.1.2.- .

Section 3.2: The Euler equation

3.2.1.- .

3.2.2.- .

Section 3.3: Regular-singular points

3.3.1.- .

3.3.2.- .

Chapter ??: The Laplace Transform

Section ??: Regular points

??.??.- .

??.??.- .

Section ??: The initial value problem

??.??.- .

??.??.- .

Section ??: Discontinuous sources

??.1.- .

??.2.- .

Section ??: Generalized sources

??.1.- .

??.2.- .

Section ??: Convolution solutions

??.1.- .

??.2.- .

Chapter 5: Systems of differential equations

Section ??: Introduction

??.1.- .

??.2.- .

Section 8.1: Systems of algebraic equations

8.1.1.- .

8.1.2.- .

Section 8.2: Matrix algebra

8.2.1.- .

8.2.2.- .

Section ??: Linear differential systems

??.??.- .

??.??.- .

Section 8.3: Diagonalizable matrices

8.3.1.- .

8.3.2.- .

Section ??: Constant coefficients systems

??.??.- .

??.??.- .

Chapter 7: Boundary value problems

Section 7.1: Eigenvalue-eigenfunction problems

7.1.1.- .

7.1.2.- .

Section 7.2: Overview of Fourier series

7.2.1.- .

7.2.2.- .

Section 7.3: Applications: The heat equation

7.3.1.- .

7.3.2.- .

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