Nyquist Criterion For Stability of Closed Loop System

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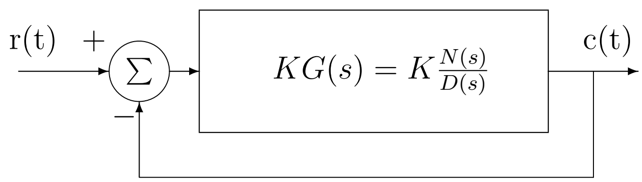
1 Nyquist Theorem

Given a closed loop system:



$$H(s) = \frac{KG(s)}{1+KG(s)} = \frac{KN(s)}{D(s)+KN(s)}$$

- n: The order of denominator polynomial $D(s)$;
- $KG(s)$ = Open Loop Transfer Function;
- $D(S)$ = Open Loop Characteristic polynomial with **O** unstable roots (unstable open loop poles) and $n - \mathbf{O}$ stable roots;
- $P(s) = D(s) + KN(s)$ = Closed Loop Characteristic polynomial with **C** unstable roots (unstable closed loop poles);
- $KG(j\omega)$ = Open Loop frequency response;
- $1 + KG(j\omega)$ = Shifted Plot.

The plot of $(1 + KG(j\omega))$ as ω varies from $\omega = -\infty$ to $\omega = \infty$ is referred to as " Nyquist Locus" in $(1 + KG(j\omega))$ plane. Plot of Nyquist Locus in $(1 + KG(j\omega))$ plane determines the stability of the Closed Loop System. Let

$$\mathbf{N} \;=\; \text{number of counter clockwise encirclements of the origin } (0+0j) \text{ by the Nyquist Locus as } \omega \text{ varies from } -\infty \text{ to } \infty, \text{ in } (1+KG(j\omega)) \text{ plane.}$$

Nyquist Theorem state:

$$\mathbf{N} = \mathbf{O} - \mathbf{C} \tag{1}$$

So this implies:

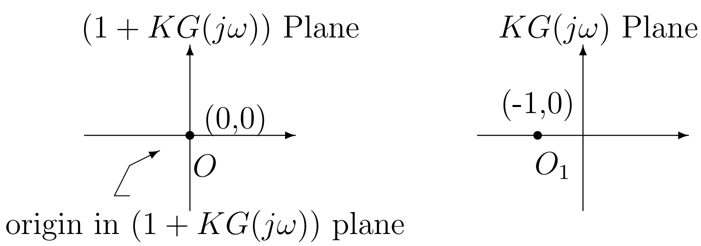
- if $\mathbf{C} = 0$, the conclusion is:

  Feedback control system is stable, if the number of counterclockwise encirclements of the origin in $(1 + KG(j\omega)$ plane is equal to the number of open loop unstable poles $\mathbf{O}$.

- if $\mathbf{O} = 0$ and $\mathbf{C} = 0$, the conclusion is:

  Feedback control system is stable if the Nyquist Locus in $(1 + KG(j\omega)$ does not encircle the origin.

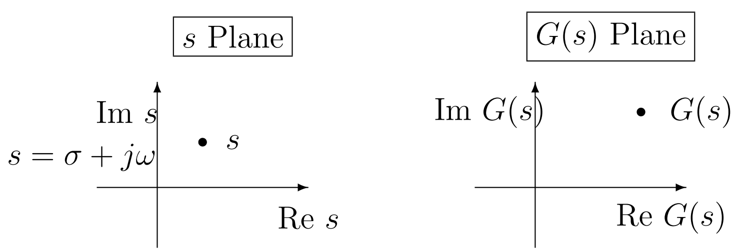
Origin in $(1 + KG(j\omega))$ transform to $-1 + j0$ point in $KG(j\omega)$ plane.



$O, (0, 0)$ in $(1 + KG(j\omega))$ plane is "Mapped" into $O_1, (-1, 0)$ in $KG(j\omega)$ plane, the point O_1 is referred to as the critical point.

2 Graphical Interpretation of $G(s)$

- Concept of $G(s)$ as a "Mapping" form s plane to $G(s)$ plane.



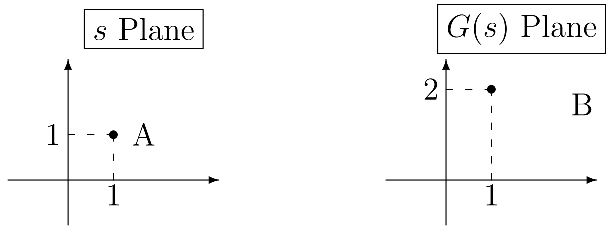
A point s in s plane is "Mapped" into a point $G(s)$ in $G(s)$ plane.

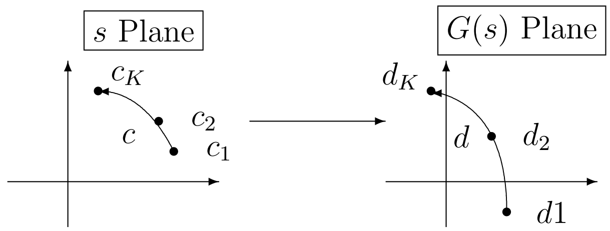
Example: Consider function $G(s) = 1 + s^2$, when $s = 1 + j$, then

$$G(s) = 1 + s^2 = 1 + (1 + j)^2 = 1 + 1 + (-1) + 2j = 1 + 2j$$

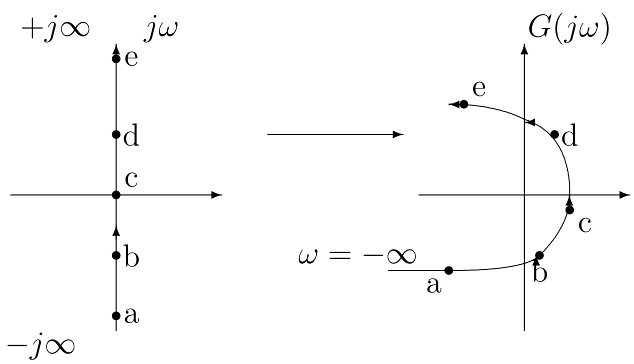
Point A in s plane has been mapped in B in $G(s)$ plane. Let s be moving on a curve c as c_1, c_2, $\cdots$, c_K, then corresponding points move in $G(s)$ plane along d_1, d_2, $\cdots$, d_K.

Thus the curve c in s plane has been mapped to curve d in $G(s)$ plane via the function $G(s)$.

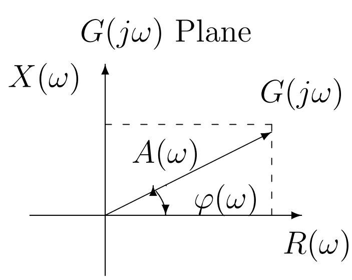




- $G(s)|_{s=j\omega} = G(j\omega)$ implies that s is allowed to take values only in the imaginary axis of the s plane. Thus as ω is varied from $\omega = -\infty$ to $\omega = +\infty$, the whole of the $j\omega$ axis in the s plane is mapped into a locus in the $G(j\omega)$ plane.



- $G(j\omega)$ can be considered as a phaser.



$$G(j\omega) = R(\omega) + jX(\omega)$$

$$G(j\omega) = R(\omega) + jX(\omega) = \left[R^2(\omega) + X^2(\omega)\right]^{\frac{1}{2}} \angle \tan^{-1} \frac{X(\omega)}{R(\omega)} = A(\omega)\angle\varphi(\omega)$$

And it has the following properties. If given $G_1(j\omega)$, $G_2(j\omega)$ as

$$\begin{aligned} G_1(j\omega) &= A_1(\omega)\angle\varphi_1(\omega) = R_1(\omega) + jX_1(\omega) \\ G_2(j\omega) &= A_2(\omega)\angle\varphi_2(\omega) = R_2(\omega) + jX_2(\omega) \end{aligned}$$

then

$$\begin{aligned} G_1(j\omega) + G_2(j\omega) &= [R_1(\omega) + R_2(\omega)] + j\,[X_1(\omega) + X_2(\omega)] \\ G_1(j\omega)G_2(j\omega) &= A_1(\omega)A_2(\omega)\angle\,[\varphi_1\omega + \varphi_2(\omega)] \\ \frac{G_1(j\omega)}{G_2(j\omega)} &= \frac{A_1(\omega)}{A_2(\omega)}\angle\,[\varphi_1\omega - \varphi_2(\omega)] \end{aligned}$$

3 Proof of Nyquist Criterion

Restate the theorem here:

Given:

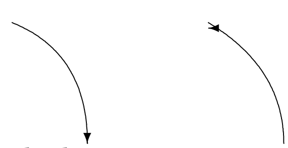
- $KG(s) =$ Open Loop Transfer Function with **O** unstable poles;
- $H(s) = \frac{KG(s)}{1+KG(s)} =$ Closed Loop Transfer Function with **C** unstable poles;
- $N =$ number of counterclockwise encirclements of $(-1+j0)$ point by the locus of $G(j\omega)$, $-\infty < \omega < +\infty$;
- $n =$ degree of $D(s)$.

Nyquist Theorem states that:

$$\mathbf{C} = -\mathbf{N} + \mathbf{O},$$

and $\mathbf{C} = 0$ implies stability of the closed loop system.

This implies that "For a system to be closed loop stable, the number of encirclements of $(-1 + j0)$ point by the locus of $G(j\omega)$, $-\infty < \omega < +\infty$ in the counterclockwise direction is equal to the number of unstable open loop poles."



clockwise counterclockwise

Proof:

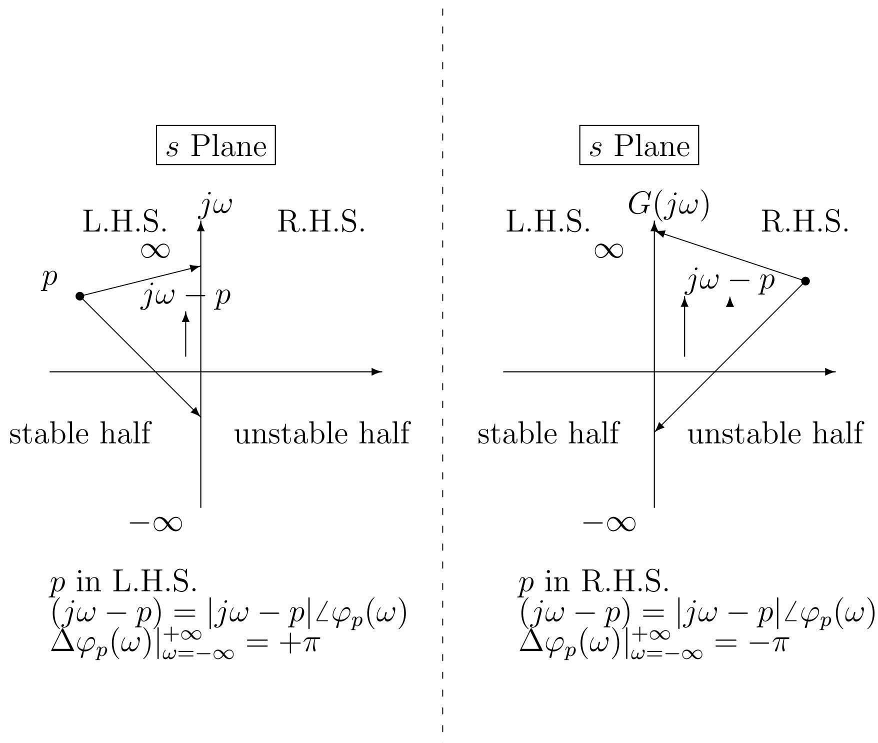
$$F(s) = 1 + KG(s) = 1 + \frac{KN(s)}{D(s)} = \frac{D(s) + KN(s)}{D(s)} = \frac{P(s)}{D(s)}$$

$$F(s) = \frac{\prod_{i=1}^{n}(s - p_{ic})}{\prod_{k=1}^{n}(s - p_{ko})}, \quad F(j\omega) = \frac{\prod_{i=1}^{n}(j\omega - p_{ic})}{\prod_{k=1}^{n}(j\omega - p_{ko})}$$

$$F(j\omega) = |F(j\omega)|\angle\varphi_F(\omega), \quad \varphi_F(\omega) = \sum_{i=1}^{n} \varphi_{ic}(\omega) - \sum_{k-1}^{n} \varphi_{ko}(\omega)$$

then

$$\Delta\varphi_F(\omega)|_{\omega=-\infty}^{+\infty} = \sum_{i=1}^{n} \Delta\varphi_{ic}(\omega)|_{\omega=-\infty}^{+\infty} - \sum_{k=1}^{n} \Delta\varphi_{ko}(\omega)|_{\omega=-\infty}^{+\infty}$$



Then for $F(s) = \frac{D(s)+KN(s)}{D(s)}$, it has n zeros and n poles. And it has zeros which are same as Closed Loop poles and has poles which are same as open loop poles.

Now

$$F(s) \text{ has } \begin{cases} (n-\mathbf{C}) \text{ zeros in L.H.S. and } \mathbf{C} \text{ zeros in R.H.S..} \\ (n-\mathbf{O}) \text{ poles in L.H.S. and } \mathbf{O} \text{ poles in R.H.S..} \end{cases}$$

Thus

$$\begin{aligned} \Delta\varphi_F(\omega)|_{\omega=-\infty}^{+\infty} &= \sum_{i=1}^{n} \Delta\varphi_{ic}(\omega)|_{\omega=-\infty}^{+\infty} - \sum_{k=1}^{n} \Delta\varphi_{ko}(\omega)|_{\omega=-\infty}^{+\infty} \\ &= [\pi(n-\mathbf{C}) - \pi\mathbf{C}] - [\pi(n-\mathbf{O}) - \pi\mathbf{O}] = -2\pi(\mathbf{C}-\mathbf{O}) = 2\pi N \qquad (2) \end{aligned}$$

where

$$\begin{aligned} N &= \text{counterclockwise encirclements of origin in } (1+G(j\omega)) \text{ plane} \\ &= \text{counterclockwise encirclements of } (-1+j0) \text{ in } G(j\omega) \text{ plane} \end{aligned}$$