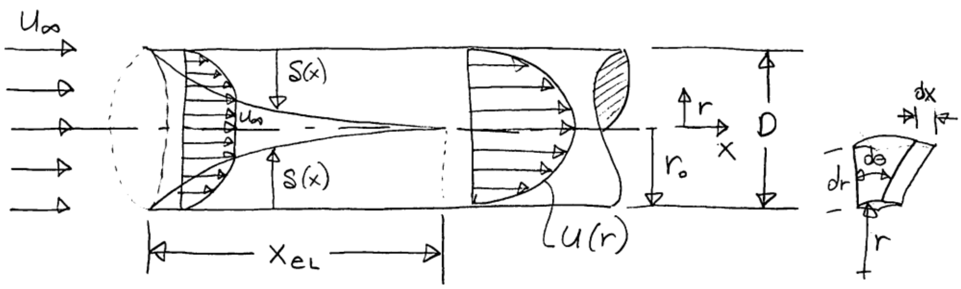
Internal Flow (Laminar) – Fully Developed Flow in Tubes

(Reorganization of Professor Nenad Miljkovic's course notes)

Define entrance length x_{el} (or developing length) as the length of tube in which velocity profile varies with radial position, r, and axial location, x.



We can estimate the magnitude of the entrance length x_{el}.

We know that the boundary layer thickness in a laminar flow on a flat plate is:

$$\frac{\delta}{x} = 5/(\mathrm{Re_x})^{0.5} \qquad \text{(Blasius solution)}$$

We can estimate that $\delta \sim \frac{D}{2}$ when the two boundary layers merge.

$$\frac{D}{2x_{el}} \sim 5.0/\left(\mathrm{Re}_{x_{el}}\right)^{0.5}$$

Note that we don't use equals (=) here because it is not a flat plate.

$$\frac{D}{x_{el}} \sim \frac{10}{\left(\mathrm{Re}_{x_{el}}\right)^{0.5}} = \frac{10}{\left(\frac{\rho U_\infty x_{el}}{\mu}\right)^{0.5}} = \frac{10}{\left(\frac{\rho U_\infty x_{el}}{\mu} * \frac{D}{D}\right)^{0.5}}$$

$$\left(\frac{D}{x_{el}}\right)^{0.5} \sim \frac{10}{(Re_D)^{0.5}}$$

Therefore $\frac{x_{el}}{D} \sim \frac{Re_D}{100} \sim 0.01 Re_D$, $\mathrm{Re_D} = \rho U_\infty D/\mu$

The actually solution experimentally verified is

$$\frac{x_{el}}{D} = 0.05 Re_D$$

Now looking at the fully developed region, with Navier-Stokes equations we get

Velocity profile in a pipe: $u(r) = r_0^2/4\mu\left(-\frac{\partial P}{\partial x}\right)\left(1-\frac{r^2}{r_0^2}\right)$

Average velocity: $\bar{u} = r_0^2/8\mu\left(-\frac{\partial P}{\partial x}\right)$ $u(r) = 2\bar{u}\left(1-\frac{r^2}{r_0^2}\right)$

Typically we want to solve for friction coefficient and pressure loss

Tube friction factor: $f = \Delta P/\left(\frac{L}{D} * \frac{1}{2} * \rho v^2\right)$

Tube friction coefficient: $C_f = \frac{\tau}{\frac{1}{2} * \rho v^2}$

Look at a finite differential element in the flow and using a force balance, we can get: $4C_f = f$.

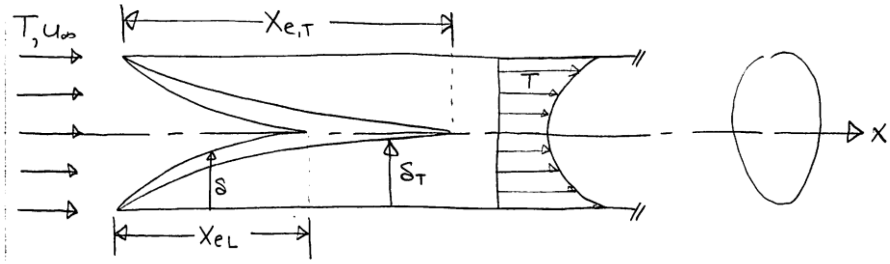
The friction factor in a pipe can be solved using the velocity profile we have achieved before:

$f = \frac{64}{Re_D}$ (Pipe friction factor for laminar flow. Also known as the Darcy-Weisbach equation)

Note that $f * Re_D$=constant for any cross section pipe.

$D_H = \frac{4A}{P}$ where A=area and P=perimeter.

Heat transfer in the pipe:



Here we have a similar situation as the hydrodynamic developing (or entrance) length but with temperature.

Thermal developing length: $\frac{x_{e,T}}{D} = 0.017Re_D Pr$

To estimate the heat transfer we can try a simple analysis

$$h = \frac{q''_{wall}}{\Delta T}$$

Since $h\Delta T \sim k_f * \frac{\Delta T}{\delta_T}$, we have $h \sim \frac{k_f}{\delta_T}$.

Assuming $\delta_T \approx \frac{r_o}{2}$ (since it is pipe flow)

$$\bar{h} \sim \frac{2k_f}{r_o} = \frac{4k_f}{D}$$

We know that $\overline{Nu_D} = \frac{\bar{h}D}{k_f} \approx 4$ just from a very simple analysis.

It is important to note here that heat transfer for internal flow problems is calculated using the bulk fluid temperature.

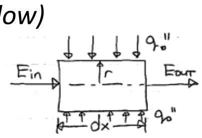
$\bar{h} = \frac{q''_{wall}}{T_w - T_b}$, T_b=bulk fluid temperature

Think of T_b as the uniform temperature of the pipe fluid if it was allowed to mix and come an equilibrium temperature in an adiabatic way.

$$T_b = \frac{1}{A\bar{V}} * \int u(r)TdA$$

Constant wall heat flux: (q''_o=constant, Fully developed flow)

Energy equation: $\rho C_p u\left(\frac{\partial T}{\partial x}\right) = k * \frac{1}{r} * \frac{\partial\left(r * \frac{\partial T}{\partial r}\right)}{\partial r}$



LHS: convection
RHS: conduction

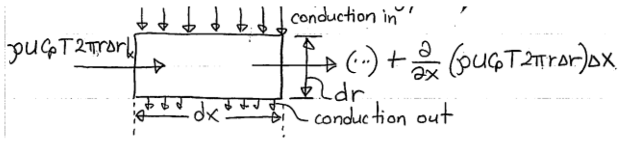
Let $\frac{T-T_b}{T_w-T_b} = f(r)$

$$q''|_{r=r_o} = -k * \frac{\partial T}{\partial r}|_{r_o} = -k * \frac{\partial f}{\partial r}|_{r_o}(T_w - T_b) = constant$$

For fully developed flow the temperature profile shape does not change: $\frac{\partial f}{\partial r}|_{r_o} = constant$

Therefore $T_w - T_b = constant$.

Solve energy equation on a fluid element:



We can get the temperature profile as:

$$T(r) = T_w - \frac{4q_o''}{k_f r_o} * \left(\frac{3r_o^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_o^2}\right)$$

Solve for T_b:

$$T_w - T_b = \frac{11}{24} * \frac{q_o'' r_o}{k_f} \quad \text{(Not a function of x)}$$

Solve for Nusselt number:

$\overline{Nu_D} = \frac{\bar{h}D}{k_f} = 4.364$ (Laminar flow in a tube with constant heat flux conditions)

We can do a similar analysis to show that for a constant wall temperature boundary condition:

$\overline{Nu_D} = \frac{\bar{h}D}{k_f} = 3.66$ (T_w = cosntant)

Our previous solution of $Nu_D \approx 4$ is pretty close to the exact analytical solutions.

Note that both of these solutions are valid only if

$Re_D = \frac{\rho \bar{u} D}{\mu} \leq 2300$ (Laminar flow in a smooth pipe, fully developed.)

Some physical insights:

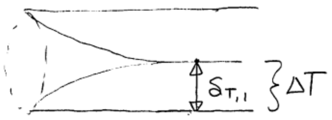
We defined the Nusselt number as the non-dimensional temperature gradient:

$-\frac{\partial \theta}{\partial n^*} = \frac{hL}{k_f} = Nu, n^* = \frac{n}{L} or\ n^* = \frac{n}{D}$ for a pipe

$$\theta = \frac{T - T_f}{T_s - T_f}$$

This tells us why it's so advantageous to go to mini or micro channel flows for cooling. By reducing the channel dimension (diameter), we reduce the effective boundary layer thickness (thermal, δ_T) and pump up the heat transfer.

Large Channel:

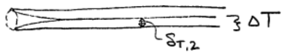


$$-\frac{\partial \Theta}{\partial n^*} = \frac{h_1 D_1}{k_f} = Nu \simeq 4$$

$$h_1 = \frac{4k_f}{D_1}$$

$\Longleftrightarrow$

Micro Channel:



$$-\frac{\partial \Theta}{\partial n^*} = \frac{h_2 D_2}{k_f} = Nu \approx 4$$

$$h_2 = \frac{4k_f}{D_2}$$

Since $D_2 \ll D_1$, $h_2 \gg h_1$

It is the direction of research these days to push dimensions to the micro length scale to maximize h. One drawback to this is the excess pumping power required to drive the fluid.

$$\Delta \mathrm{P} = \mathrm{f} * \frac{\mathrm{L}}{\mathrm{D}} * \frac{1}{2} * \rho \bar{V}^2$$

As D decreases, $\Delta \mathrm{P}$ increases non-linearly since $\mathrm{f} = \frac{64}{\mathrm{Re_D}} = \frac{64\mu}{\rho \bar{V} D}$ and $\bar{V} \propto D^{-2}$ for a constant mass flow rate.