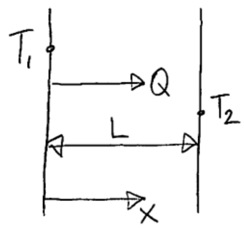
One-Dimensional, Steady-State Heat Conduction
(Reorganization of the Lecture Notes from Professor Nenad Miljkovic)

1-D, steady state, $\dot{Q}''' = 0$, k=constant

We know from heat diffusion equation that $\nabla^2 T = 0$.

(1) Slab



$$\int \frac{\partial^2 T}{\partial x^2} = \int 0$$
$$\int \frac{\partial T}{\partial x} = \int C_1$$
$$\mathrm{T(x)} = \mathrm{C}_1 x + C_2$$

Given the boundary conditions $\mathrm{T(x=0)} = \mathrm{T}_1, \mathrm{T(x=L)} = \mathrm{T}_2$

$$\mathrm{T(x)} = \frac{\mathrm{T}_2 - \mathrm{T}_1}{L} * x + T_1$$

So if we define a heat transfer resistance: R, A=area

$$\mathrm{Q} = \frac{\Delta \mathrm{T}}{\mathrm{R}}$$
$$\mathrm{Q} = -\mathrm{kA}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}$$

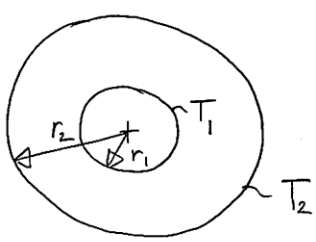
From above we know that $\frac{\partial \mathrm{T}}{\partial \mathrm{x}} = \frac{\mathrm{T}_2 - \mathrm{T}_1}{\mathrm{L}} = -\frac{\Delta T}{L}$

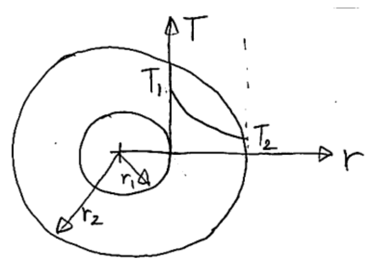
$$\mathrm{Q} = \frac{\mathrm{kA}}{\mathrm{L}} * \Delta\mathrm{T} = \frac{1}{\mathrm{R}} * \Delta\mathrm{T}$$

Thermal resistance of a solid slab:

$$\mathrm{R}_{\mathrm{slab}} = L/kA$$

(2) Circular cylinder





We know the heat equation in radial coordinate is:

$$\frac{1}{\mathrm{r}} * \frac{\partial\left(\mathrm{r} * \frac{\partial \mathrm{T}}{\partial \mathrm{r}}\right)}{\partial \mathrm{r}} = 0$$
$$r\frac{\partial \mathrm{T}}{\partial \mathrm{r}} = C_1$$
$$\mathrm{T(r)} = \mathrm{C}_1 \ln(r) + C_2$$

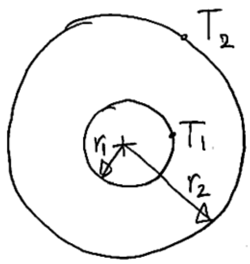
After applying the boundary conditions:

$$\frac{T - T_1}{T_2 - T_1} = \ln\left(\frac{r}{r_1}\right) / \ln(\frac{r_2}{r_1})$$

Similarly, cylindrical thermal resistance:

$$R_{cyl} = |\ln(r_2/r_1)/2\pi kL|$$

(3) Spherical system



Spherical heat diffusion equation:

$$\frac{1}{r^2} * \frac{\partial\left(r^2 * \frac{\partial T}{\partial r}\right)}{\partial r} = 0$$

Spherical temperature profile:

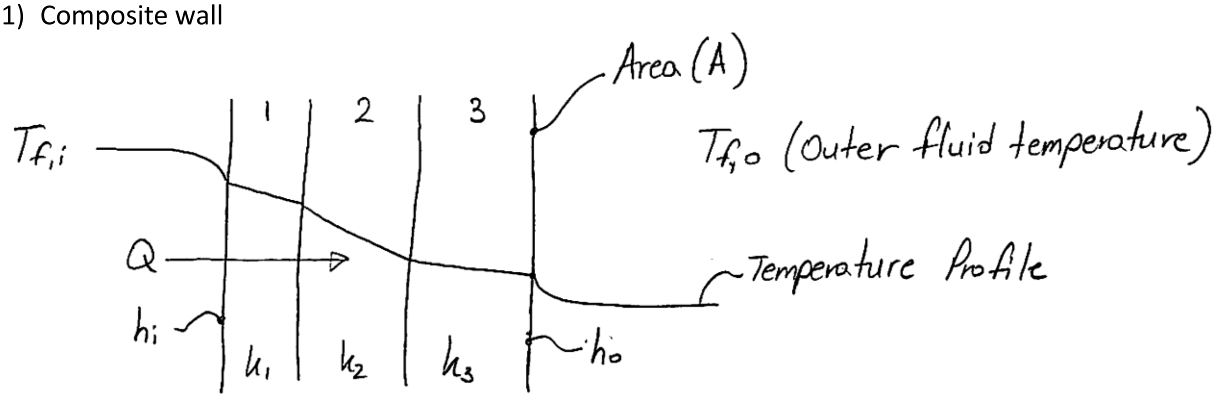
$$\frac{T - T_2}{T_1 - T_2} = (\frac{1}{r} - \frac{1}{r_2}) / (\frac{1}{r_1} - \frac{1}{r_2})$$

Spherical thermal resistance:

$$R_{sph} = \frac{\left|\frac{1}{r_1} - \frac{1}{r_2}\right|}{4\pi k}$$

Composite problems

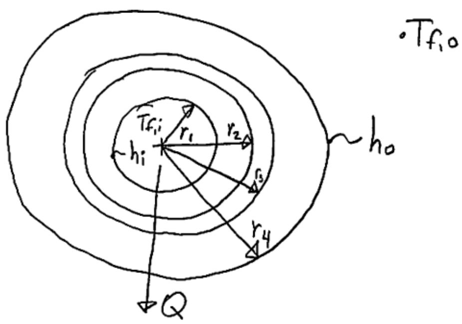
(1) Composite wall



$$Q = \frac{A(T_{f,i} - T_{f,o})}{\frac{1}{h_i} + \frac{1}{h_o} + \sum_{i=1}^{n} \frac{L_i}{k_i}}$$

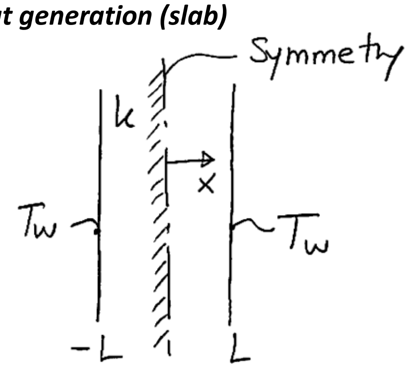
L_i=thickness of the i-th wall
k_i=thermal conductivity of the i-th wall
h_i=inner wall heat transfer coefficient
h_o=outer wall heat transfer coefficient

(2) Composite cylinder



$$Q = \frac{2\pi l\left(T_{f,i} - T_{f,o}\right)}{\frac{1}{h_i r_i} + \frac{1}{h_o r_{n+1}} + \sum_{j=1}^{n} \frac{\ln\left(\frac{r_{i+1}}{r_i}\right)}{k_i}}$$

Heat generation (slab)



Steady state, 1-D, constant properties, uniform $\dot{Q}'''$.

$$\frac{\partial\left(k * \frac{\partial T}{\partial x}\right)}{\partial x} + \dot{Q}''' = \frac{\rho C_p \partial T}{\partial t} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{Q}'''}{k} = 0$$

Boundary conditions:

$$T(x = L) = T_w$$

$$\frac{\partial T}{\partial x}|_{x=0} = 0$$

$$T(x) = T_w + \frac{\dot{Q}''' L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

$T_{max} = T_w + \frac{\dot{Q}''' L^2}{2k}$ at x=0.

$$q''_{out} = \dot{Q}''' L$$

In general, for heat generation we have the following formulation:

$$-\frac{k}{r^n}\frac{\partial\left(r^n * \frac{\partial T}{\partial r}\right)}{\partial r} = \dot{Q}'''$$

Where n=0 for slab, n=1 for cylinder, n=2 for sphere.
This only works if $\dot{Q}''' = \text{constant}$
Our generalized result is:
Temperature profile:

$$T(r) = T_a + \frac{\dot{Q}''' a^2}{2(n+1)k}\left(1 - \frac{r^2}{a^2}\right)$$

Heat flux:

$$q'' = \frac{\dot{Q}'''r}{n+1}$$

Maximum temperature where T_a is the boundary temperature on the outside:

$$T_{max} = T(r=0) = T_a + \frac{\dot{Q}'''a^2}{2(n+1)k}$$

Conduction for general shapes

No convection, steady state, $\dot{Q}''' = 0$, k=consant, 1-D

Define an orthogonal coordinate system:

u_1, u_2, u_3 (directions) s_1, s_2, s_3 (lengths) $ds_1 = h_1 du_1$, $ds_2 = h_2 du_2$, $ds_3 = h_3 du_3$

$$q'' = -k * \frac{\partial T}{\partial s_1}$$

$$Q = Aq'' = -kA(u_1)\frac{\partial T}{\partial s_1}$$

Rearranging and integrating (assuming $ds_1 = du_1$)

$$\frac{Q}{k} * \int_{(u_1)_a}^{(u_1)_b} \frac{du_1}{A(u_1)} = -\int_{T_a}^{T_b} dT = T_a - T_b = \Delta T$$

$$Q = \Delta T / (\frac{1}{k} * \int \frac{du_1}{A(u_1)}) = \Delta T / R$$

$$R = \frac{1}{k} * \int \frac{du_1}{A(u_1)})$$

Double check our previous solutions:

(1) Slab: $u_1 = x, du_1 = dx, A(u_1) = constant$

$$R_{slab} = \frac{1}{kA}\int du_1 = \frac{1}{kA}\int dx = L/kA$$

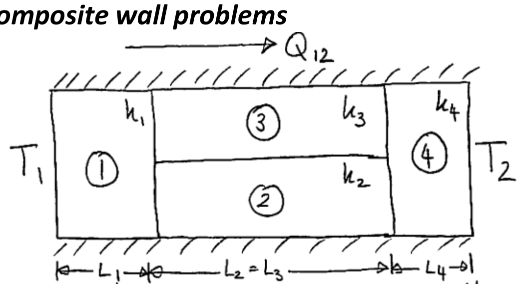
(2) Cylinder: $u_1 = r$, $du_1 = dr$, $A(u_1) = A(r) = 2\pi rl$

$$R_{cyl} = \frac{1}{k} * \int_{r_1}^{r_2} \frac{dr}{2\pi rl} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi kl}$$

(3) Sphere: $u_1 = r, du_1 = dr, A(u_1) = A(r) = 4\pi r^2$

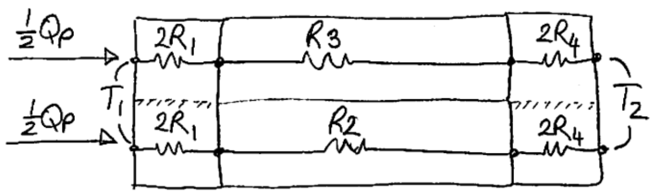
$$R_{sph} = \frac{1}{k} * \int_{r_1}^{r_2} \frac{dr}{4\pi r^2} = \frac{1}{4\pi k} * \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{1}{4\pi k}(1\backslash r_1 - 1\backslash r_2)$$

Composite wall problems



There are two ways to analyze this problem. They give different answers.

(1) Parallel approach



Add adiabatic surface to divide R_1 and R_4 into two.

$$R_{Lower} = \frac{L_1}{k_1 A_2} + \frac{L_2}{k_2 A_2} + \frac{L_4}{k_4 A_2}$$

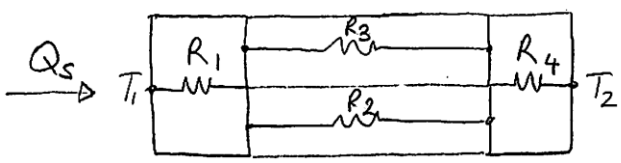
$$R_{Upper} = \frac{L_1}{k_1 A_3} + \frac{L_2}{k_3 A_3} + \frac{L_4}{k_4 A_3}$$

$$\frac{1}{R_p} = \frac{1}{R_{Lower}} + \frac{1}{R_{Upper}}$$

$$\frac{1}{R_p} = \left(\frac{A_2}{\frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_4}{k_4}} + \frac{A_3}{\frac{L_1}{k_1} + \frac{L_3}{k_3} + \frac{L_4}{k_4}} \right)$$

$$Q_p = \frac{\Delta T}{R_p} == \left(\frac{A_2}{\frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{L_4}{k_4}} + \frac{A_3}{\frac{L_1}{k_1} + \frac{L_3}{k_3} + \frac{L_4}{k_4}} \right) \Delta T$$

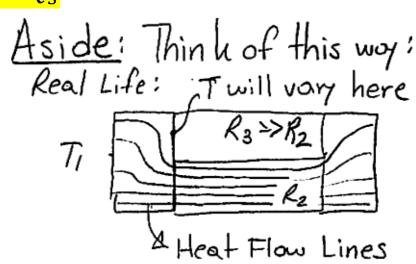
(2) Series approach



$$R_s = \frac{L_1}{k_1 A_1} + \left(\frac{k_2 A_2}{L_2} + \frac{k_3 A_3}{L_2} \right)^{-1} + \frac{L_4}{k_4 A_1}$$

$$Q_s = \frac{\Delta T}{R_s} = \left(\frac{L_1}{k_1 A_1} + \left(\frac{k_2 A_2}{L_2} + \frac{k_3 A_3}{L_2} \right)^{-1} + \frac{L_4}{k_4 A_1} \right)^{-1} \Delta T$$

It is always true that $Q_p \le Q_{real} \le Q_s$



While by adding infinite conductance between R1 and R2, R3, the temperature at the boundary becomes the same.

In general:

1. Adding adiabatic surface: R goes up, Q goes down;
2. Allowing infinite conductance: R goes down, Q goes up.