The Basics of Heat Conduction

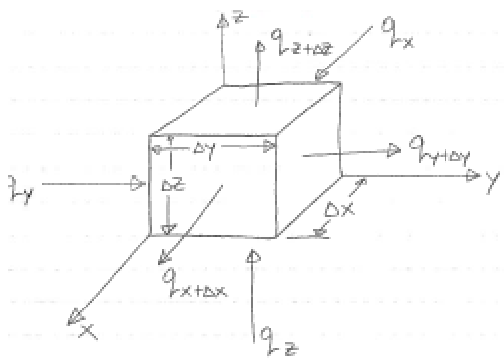
Conduction may be viewed as the transfer of energy from the more energetic to the less energetic particles of a substance due to interaction between the particles.

It is possible to quantify heat transfer processes in terms of appropriate rate equation. These equations may be used to compute the amount of energy being transferred per unit time. For heat conduction, the rate equation is known as Fourier's law. For the one-dimensional plane wall having a temperature distribution T(x), the rate equation is expressed as

$$q''_x = -k * \frac{dT}{dx}$$

The heat flux q''_x ($\frac{W}{m^2}$) is the heat transfer rate in the x direction per unit area perpendicular to the direction of transfer, and it is proportional to the temperature gradient, dT/dx, in this direction. The parameter k is a transport property known as the thermal conductivity (W/mK) and is a characteristic of the wall material. The minus sign is a consequence of the fact that heat is transferred in the direction of decreasing temperature.

Heat Diffusion Equation



From 1st law of thermodynamics

$$\dot{Q}_{in} - \dot{Q}_{out} + \dot{Q}_{gen} = \dot{Q}_{ST}$$

Entering and leaving through the surface are

$$\dot{Q}_{in} = q_x A_x + q_y A_y + q_z A_z$$
$$\dot{Q}_{out} = q_{x+\Delta x} A_{x+\Delta x} + q_{y+\Delta y} A_{y+\Delta y} + q_{z+\Delta z} A_{z+\Delta z}$$

For the Cartesian coordinate system

$$A_x = A_{x+\Delta x} = \Delta y \Delta z$$
$$A_y = A_{y+\Delta y} = \Delta x \Delta z$$
$$A_z = A_{z+\Delta z} = \Delta x \Delta y$$

Within the control volume, we have energy generation $\dot{Q}'''$

$$\dot{Q}_{gen} = \dot{Q}''' V = \dot{Q}''' \Delta x \Delta y \Delta z$$

Within the control volume, the change in stored energy is

$$\dot{Q}_{ST} = \frac{\partial U}{\partial t} = \frac{\partial (Mu)}{\partial t} = \frac{\partial (\rho V u)}{\partial t}$$

Assuming a constant density ρ

$$\dot{Q}_{ST} = \rho \frac{\partial u}{\partial t} \Delta x \Delta y \Delta z$$

Adopting an equation of state (simple incompressible substance with constant specific heat)

$$\dot{Q}_{ST} = \rho C_p \frac{\partial T}{\partial t} \Delta x \Delta y \Delta z$$

Now we plug everything together, using Taylor series expansion and neglecting higher order terms:

$$\dot{Q}_{in} - \dot{Q}_{out} = -\Delta x \Delta y \Delta z \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)$$

As mentioned before, $q_x = -\frac{k\,\partial T}{\partial x}$

Substituting the constitutive equations into the first law:

$$\frac{\partial\left(k\frac{\partial T}{\partial x}\right)}{\partial x} + \frac{\partial\left(k\frac{\partial T}{\partial y}\right)}{\partial y} + \frac{\partial\left(k\frac{\partial T}{\partial z}\right)}{\partial z} + \dot{Q}''' = \rho C_p \frac{\partial T}{\partial t}$$

This second order PDE is the conservation of thermal energy for an isotropic incompressible substance with density and specific heat independent of time.

If thermal conductivity does not depend on location

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{Q}'''}{k} = \frac{\rho C_p}{k} * \frac{\partial T}{\partial t} = \frac{1}{\alpha} * \frac{\partial T}{\partial t}$$