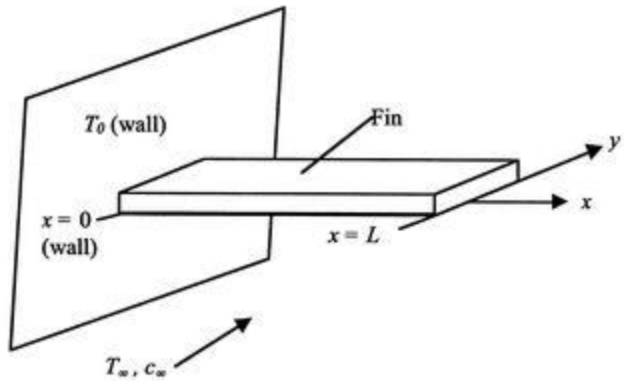
Fins

(Reorganization of the Lecture Notes from Professor Nenad Miljkovic and Professor Z. S. Spakovszky)

Quasi-1D fin conduction



Assumptions:

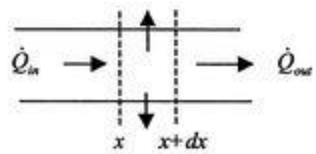
1. Perimeter P and cross-sectional area A of the fin are constant
2. No heat generation
3. The heat transfer is 1-D
4. The heat transfer coefficient for the fin is known and has the value h. The end of the fin can have a different heat transfer coefficient, which we can call $\mathrm{h_L}$.

Note: assumption 3 means that there is no temperature variation in the y-direction. Temperature in the fin will be assumed to be a function of x only.

This may seem a drastic simplification, and it needs some explanation. With a fin cross-section equal to A and a perimeter P , the characteristic dimension in the transverse direction is A/P (For a circular fin, for example, $\frac{A}{P} = r/2$). The regime of interest will be taken to be that for which the Biot number is much less than unity, $Bi = \frac{h(A/P)}{k} \ll 1$, which is a realistic approximation in practice.

The physical content of this approximation can be seen from the following. Heat transfer per unit area out of the fin to the fluid is roughly of magnitude $\sim h(T_w - T_\infty)$ per unit area. The heat transfer per unit area within the fin in the transverse direction is (again in the same approximate terms) $\frac{k(T_i - T_w)}{A/P}$ where T_i is an internal temperature (temperature in the center of the fin). These two quantities must be of the same magnitude. If $\frac{h(A/P)}{k} \ll 1$, then $(T_i - T_w)/(T_w - T\_\infty) \ll 1$. In other words, if $Bi \ll 1$, there is a much larger capability for heat transfer per unit area across the fin than there is between the fin and the fluid, and thus little variation in temperature inside the fin in the transverse direction. To emphasize the point, consider the limiting case of zero heat transfer to the fluid, i.e., an insulated fin. Under these conditions, the temperature within the fin would be uniform and equal to the wall temperature.

If there is little variation in temperature across the fin, an appropriate model is to say that the temperature within the fin is a function of x only, $T = T(x)$, and use a quasi-one-dimensional approach.



Element of fin showing heat transfer

Consider an element, dx, of the fin as shown above. There is heat flow of magnitude $\dot{Q}_{in}$ at the left-hand side and heat flow out of magnitude which can be approximated as $\dot{Q}_{out} = \dot{Q}_{in} + \frac{d\dot{Q}}{dx} * dx$ at the right hand side. There is also heat transfer around the perimeter on the top, bottom, and sides of the fin. From a quasi-one-dimensional point of view, this is a situation similar to that with internal heat sources, but here, for a cooling fin, in each elemental slice of thickness dx there is essentially a heat sink of magnitude $\mathrm{Pdxh}(\mathrm{T} - \mathrm{T}_\infty)$, where Pdx is the area for heat transfer to the fluid.

The heat balance for the element in Figure 18.4 can be written in terms of the heat flux using $\dot{Q} = \dot{q}A$, for a fin of constant area:

$$\dot{q}A = Ph(T - T_\infty)dx + (\dot{q}A + \frac{d\dot{q}}{dx} * dx\ A)$$

$$\frac{\mathrm{A}d\dot{q}}{dx} + Ph(T - T_\infty) = 0$$

In terms of the temperature distribution in the fin, $\mathrm{T(x)}$:

$$\frac{\mathrm{d}^2T}{dx^2} - \frac{Ph}{Ak}(T - T_\infty) = 0$$

Define $(\mathrm{T} - \mathrm{T}_\infty)/(T_0 - T_\infty)$ as $\Delta\tilde{T}$, where the values of $\Delta\tilde{T}$ range from zero to one according to the boundary conditions (T_0 and T_∞ are wall and ambient fluid temperature respectively). Also define $\varepsilon = \mathrm{x/L}$, where ε also ranges over zero to one.

$$\frac{\mathrm{d}^2\Delta\tilde{T}}{d\varepsilon^2} - \frac{Ph}{kA} * L^2\Delta\tilde{T} = 0$$

Define m by $\mathrm{m}^2 = \frac{Ph}{kA}$

$$\frac{\mathrm{d}^2\Delta\tilde{T}}{d\varepsilon^2} - m^2L^2\Delta\tilde{T} = 0$$

The second order equation has the solution

$$\Delta\tilde{T} = ae^{mL\varepsilon} + be^{-mL\varepsilon}$$

i. Insulated tip case

At $\varepsilon = 0$, $\Delta\tilde{T}(0) = a + b = 1$.

At $\varepsilon = 1$ the temperature gradient is zero assuming that there is negligible heat transfer at the tip.

$$\frac{\mathrm{d}\Delta\tilde{T}}{d\varepsilon}(L) = mLae^{mL} - mLbe^{-mL} = 0$$

Therefore,

$$\Delta\tilde{T} = (\cosh mL(1-\varepsilon))/(\cosh mL)$$

$(\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}, \tanh x = \sinh x / \cosh x = \frac{e^x - e^{-x}}{e^x + e^{-x}})$

In terms of the actual heat transfer parameters it is written as

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{cosh\left(\left(1-\frac{x}{L}\right)\sqrt{\frac{hP}{kA}}L\right)}{cosh\left(\sqrt{\frac{hP}{kA}}L\right)}$$

The amount of heat removed from the wall due to the fin is:

$$\dot{Q} = -kA\frac{d(T-T_\infty)}{dx}|_{x=0} = \tanh(mL)\sqrt{kAhP}(T_0 - T_\infty)$$

ii. Infinite fin case ($x \to \infty$)

Boundary conditions: $\Delta\tilde{T}(x=0) = 1, \Delta\tilde{T}(x = L \to \infty) = 0$

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-\frac{hP}{kA}*x}$$

$$\dot{Q} = -kA\frac{d(T-T_\infty)}{dx}|_{x=0} = \sqrt{kAhP}(T_0 - T_\infty)$$

iii. Prescribed tip temperature ($T(L) = T_L$)

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{\frac{T_L - T_\infty}{T_0 - T_\infty}\sinh\left(\sqrt{\frac{hP}{kA}}x\right) + sinh\left(\left(1-\frac{x}{L}\right)\sqrt{\frac{hP}{kA}}L\right)}{\sinh\left(\sqrt{\frac{hP}{kA}}L\right)}$$

$$\dot{Q} = -kA\frac{d(T-T_\infty)}{dx}|_{x=0} = \sqrt{kAhP}(T_0 - T_\infty) * (\cosh(mL) - \frac{T_L - T_\infty}{T_0 - T_\infty})/\sinh(mL)$$

iv. Convective heat transfer at the tip ($h_L\Delta\tilde{T} = -kd\Delta\tilde{T}/dx|_{x=L}$)

$$\frac{T - T_\infty}{T_0 - T_\infty} = \frac{cos\,h\left(\sqrt{\frac{hP}{kA}}(L-x)\right) + \left(\frac{h_L}{mk}\right) * sinh\left(\left(1-\frac{x}{L}\right)\sqrt{\frac{hP}{kA}}L\right)}{\cosh\left(\sqrt{\frac{hP}{kA}}L\right) + \left(\frac{h_L}{mk}\right) * \sinh\left(\sqrt{\frac{hP}{kA}}L\right)}$$

$$\dot{Q} = -kA\frac{d(T-T_\infty)}{dx}|_{x=0} = \sqrt{kAhP}(T_0 - T_\infty) * \frac{\sinh(mL) + \left(\frac{h_L}{mk}\right)\cosh(mL)}{\cosh(mL) + \left(\frac{h_L}{mk}\right)\sinh(mL)}$$

Fin efficiency (Insulated tip)

Fin efficiency

$$\eta_{fin} = \frac{Actual\ heat\ transfer}{\text{Heat transfer if } T = T_0 \text{ everywhere}} = q_{actual}/q_{ideal}$$

$$\eta_{fin} = \tanh(mL)/mL$$

Fin resistance (R_{fin}) for insulated tip case

$$\mathrm{R_{fin}} = \frac{T_0 - T_\infty}{q_{actual}} = \frac{1}{\sqrt{kAhP}\,\tanh(mL)} = 1/hA_{fin}\eta_{\mathrm{fin}}$$

Where $\mathrm{A_{fin}}$ = outside area of fin (PL), η_{fin}= fin efficiency, h=heat transfer coefficient.

Fin effectiveness (ε_{fin})

$$\varepsilon_{\mathrm{fin}} = \frac{actual\ heat\ transfer}{heat\ transfer\ if\ no\ fin} = \frac{\mathrm{q_{ideal}} * \eta_{\mathrm{fin}}}{q_{ideal}\left(\frac{A}{PL}\right)} = \frac{\eta_{\mathrm{fin}}}{\left(\frac{A}{PL}\right)}$$

Where P=perimeter, L is the fin length and A is the cross-sectional area of fin.