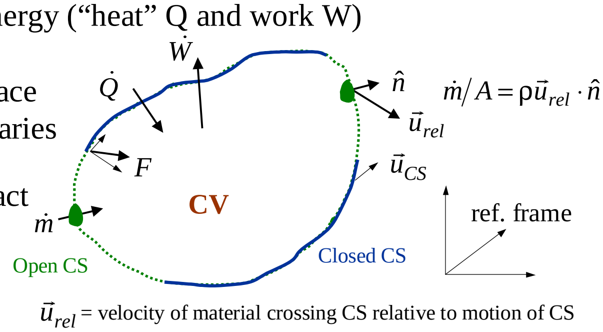
Conservation Equations

- Govern mechanics and thermodynamics of systems
- Control Mass Laws
  - mass: can not create or destroy mass (e.g., neglect nuclear reactions)
  - momentum: Newton's Law, F=ma
  - energy: 1st Law of thermodynamics, $dE=\delta Q- \delta W$
  - entropy: 2nd Law, $dS=\delta Q/T+\delta P_s$
- Propulsion systems generally employ fluid flow
  - need to write conservation laws in terms for **Control Volumes**

Reynolds Transport Theorem - RTT

- Provides general form for converting **conservation laws** from control mass to **control volumes**
- Take arbitrary control volume (**CV**); can be moving
  - mass and energy ("heat" Q and work W) can cross control surface (**CS**) boundaries
  - forces also act on CS and mass in CV



$\vec{u}_{rel}$ = velocity of material crossing CS relative to motion of CS

RTT Equation

- Take any **extensive property** **B**, that follows a "conservation" law and its **intensive version** β (per mass); can show

$$\boxed{\left.\frac{dB}{dt}\right|_{CM} = \frac{d}{dt}\int_{CV}\rho\beta\, dV + \int_{CS}\rho\beta(\vec{u}_{rel}\cdot\hat{n})\,dA}$$

Replace with appropriate Control Mass Conservation Law

Storage term (rate of increase inside CV)

Net flux of property leaving CV, carried by flow (outflow - inflow)

- Always leads to **PICO** relationship

Production + **I**nput = **C**hange (in time) + **O**utput

Mass Conservation

- If property of interest is mass

$$B = m,\quad \beta = \frac{dB}{dm} = \frac{d(m)}{dm} = 1$$

- From **RTT**

$$\left.\frac{dB}{dt}\right|_{CM} = \frac{d}{dt}\int_{CV}\rho\beta\, dV + \int_{CS}\rho\beta(\vec{u}_{rel}\cdot\hat{n})\,dA$$

$$\left.\frac{d(m)}{dt}\right|_{CM} = \frac{d}{dt}\int_{CV}\rho 1\, dV + \int_{CS}\rho 1(\vec{u}_{rel}\cdot\hat{n})\,dA$$

- **CM:** $\left.\frac{d(m)}{dt}\right|_{CM} = 0$

Integral Control Volume Form of Mass Conservation

- **CV:**

$$0 = \frac{d}{dt}\int_{CV}\rho\, dV + \int_{CS}\rho(\vec{u}_{rel}\cdot\hat{n})\,dA$$

Simplified Mass Conservation

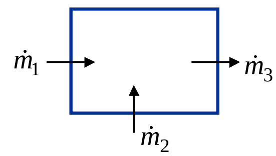
$$0=\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho dV+\int_{CS}\rho(\vec{u}_{rel}\cdot\hat{n})dA$$

- **Uniform flow (at CS)** - no variations across flow

$$\int_{CS(t)}\rho(\vec{u}_{rel}\cdot\hat{n})dA=\sum_{outlets}\rho u_{rel}A-\sum_{inlets}\rho u_{rel}A$$
$$=\sum_{outlets}\dot{m}-\sum_{inlets}\dot{m}$$

- Add **Steady-State**

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho dV=0$$



$$\Rightarrow\sum_{outlets}\dot{m}=\sum_{inlets}\dot{m}$$

$$\dot{m}_1+\dot{m}_2=\dot{m}_3$$

$\cancel{P}$I=$\cancel{C}$O $\rightarrow$ **I**nput = **O**utput

Conservation of Momentum

- **Linear momentum**

$$\vec{\mathrm{B}}=m\vec{u},\quad\vec{\beta}=\vec{u}$$

- **RTT** then gives

$$\left.\frac{\mathrm{d}(m\vec{u})}{\mathrm{d}t}\right|_{CM}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho\vec{u}dV+\int_{CS}\rho\vec{u}(\vec{u}_{rel}\cdot\hat{n})dA$$

- Use **Newton's Law**

"production" of momentum

$$\left.\frac{\mathrm{d}(m\vec{u})}{\mathrm{d}t}\right|_{CM}=\sum\vec{F}=\dot{\vec{P}}_{momentum}$$

total force acting *on fluid*

$$\sum\vec{F}_{CV}=\dot{\vec{P}}_{momentum}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho\vec{u}dV+\int_{CS}\rho\vec{u}(\vec{u}_{rel}\cdot\hat{n})dA$$

Production = **C**hange + **O**ut - **I**n

Force Terms

- Examine different forces that can act on matter in our control volume

e.g., gravity e.g., pressure, shear,...

$$\sum \vec{F}_{on\,CV} = \sum_{on\,CV} \vec{F}_{body} + \sum_{on\,CS} \vec{F}_{surface}$$

- **Body forces**

$$\vec{F}_{body} = \int_{CV} \rho \vec{f} dV \quad \text{with } \vec{f} = \text{body force/mass,}$$

i.e, **acceleration**

- **Surface forces**
  - **free surfaces**: not connected to solid body crossing control surface
  - **connected surfaces**: solid boundaries where there are reaction forces

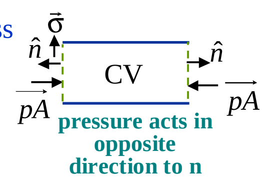
Surface Forces

- **Free surfaces**

normal stress shear stress

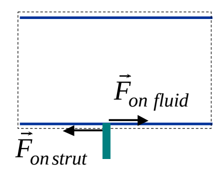
$$\vec{F}_{surface} = -\int_{free} p\hat{n} dA + \int_{free} \vec{\sigma}_{shear} dA$$

0 if inviscid



- **Connected solid surfaces**
  - force on fluid is reaction force (inverse) of force on solid body

$$\vec{F}_{on\,fluid} = -\vec{F}_{on\,solid\,body}$$



- Combine

Integral Control Volume Form of Momentum Conservation

$$\vec{F}_{\substack{solid\,body\\ on\,fluid}} - \int_{open} p\hat{n} dA + \int_{open} \vec{\sigma}_{shear} dA + \int_{CV} \rho \vec{f} dV = \frac{d}{dt}\int_{CV} \rho \vec{u} dV + \int_{CS} \rho \vec{u}\left(\vec{u}_{rel} \cdot \hat{n}\right) dA$$

Conservation of Energy

- **Energy:** microscopic + macroscopic forms of energy

internal energy

$$B = E_o = E + E_{kinetic} = E + \frac{1}{2}mu^2$$

energy per mass $\rightarrow$ $\beta = e_o = e + \frac{u^2}{2}$

- **RTT** then gives

$$\left.\frac{d(E_{tot})}{dt}\right|_{CM} = \frac{d}{dt}\int_{CV} \rho e_{tot} dV + \int_{CS} \rho e_{tot}(\vec{u}_{rel} \cdot \hat{n}) dA$$

- Use 1st Law Thermodynamics for energy conservation of control mass

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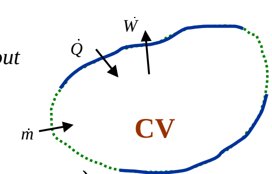
1st Law of Thermodynamics

- Differential form $dE_{CM} = \delta Q_{in} - \delta W_{out}$

$$\left.\frac{dE}{dt}\right|_{CM} = \frac{\delta Q_{in}}{dt} - \frac{\delta W_{out}}{dt} = \dot{Q}_{in} - \dot{W}_{out}$$



- Into RTT

$$\dot{Q}_{in} - \dot{W}_{out} = \frac{d}{dt}\int_{CV} \rho e_o dV + \int_{CS} \rho e_o(\vec{u}_{rel} \cdot \hat{n}) dA$$

- But work is related to forces (F·x) acting on CV
  - already examined some kinds of forces
  - let's relate them to work terms

Work and Forces

- Relationship

$$\dot{W} = \vec{F}\cdot\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{F}\cdot\vec{u}$$

Since we let work be positive when done **BY** fluid

- **Body forces**

$$\dot{W}_{body} = -\int_{CV} \rho\vec{f}\cdot\vec{u}\,dV$$

- **Fluid forces** (stresses)

$$\dot{W}_{press} = \int_{CS\ open} p(\vec{u}\cdot\hat{n})dA$$

"Flow Work"

$$\dot{W}_{shear} = -\int_{CS\ open} \vec{\sigma}\cdot\vec{u}\,dA$$

- **Reaction Forces**
  - lump into "useful" work term, e.g., shaft work

$$\dot{W}_{shaft}$$

Energy Conservation

- Combine into RTT result (*neglect shear forces*)

$$\dot{Q}_{in} - \dot{W}_{shaft} + \int_{CV}\rho\vec{f}\cdot\vec{u}\,dV - \int_{\substack{CS\\open}} p(\vec{u}\cdot\hat{n})dA = \frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho e_o dV + \int_{CS}\rho e_o(\vec{u}_{rel}\cdot\hat{n})dA$$

Similar Form

Combine flow work and energy flux

$$\rho e_o + \rho\frac{p}{\rho} = \rho\left(e_o + \frac{p}{\rho}\right) = \rho h_o$$

Stagnation Enthalpy

$$h_o = h + \frac{u^2}{2}$$

$$\dot{Q}_{in} - \dot{W}_{shaft} + \int_{CV}\rho\vec{f}\cdot\vec{u}\,dV - \int_{\substack{CS\\open}} p(\vec{u}-\vec{u}_{rel})\cdot\hat{n}\,dA = \frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho e_o dV + \int_{CS}\rho h_o(\vec{u}_{rel}\cdot\hat{n})dA$$

- For reference frame moving with control volume at *constant velocity and no body forces*

$$\dot{Q}_{in} - \dot{W}_{shaft} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\rho e_o dV + \int_{CS}\rho h_o(\vec{u}\cdot\hat{n})dA$$

$\overset{0}{\cancel{\text{P}}}$I=CO **I**n - **O**ut **C**hange **O**ut - **I**n, of energy in mass