Boundary Layer

(Reorganization of the Lecture Notes from Professor Anthony Jacobi and Professor Nenad Miljkovic)

Consider a steady flow of a Newtonian, Fourier-Biot fluid over a flat surface with constant properties, incompressible, 2-D flow and no body force.

Note:

1. A Newtonian fluid is a fluid in which the viscous stresses arising from its flow, at every point, are linearly proportional to the local strain rate-the rate of change of its deformation over time. That is equivalent to saying that those forces are proportional to the rates of change of the fluid's velocity vector as one moves away from the point in question in various directions.
2. A non-Newtonian fluid is a fluid whose viscosity is variable based on applied stress or force. Many polymer solutions and molten polymers are non-Newtonian fluids, as are many commonly found substances such as ketchup, starch suspensions, paint, and shampoo. In a Newtonian fluid, the relation between the shear stress and the strain rate is linear, the constant of proportionality being the coefficient of viscosity. In a non-Newtonian fluid, the relation between the shear stress and the strain rate is nonlinear, and can even be time-dependent. Therefore a constant coefficient of viscosity cannot be defined.

Steady flow: $\frac{\partial}{\partial t} = 0$

2-D flow: $w = 0, \frac{\partial}{\partial z} = 0$

Therefore for the continuity equation what we are left with is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

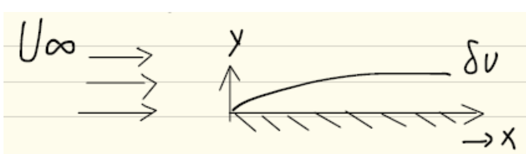
x-momentum

$$\rho\left(\frac{\partial u}{\partial t} + u * \frac{\partial u}{\partial x} + v * \frac{\partial u}{\partial y} + w * \frac{\partial u}{\partial z}\right) = \rho f_x - \frac{\partial P}{\partial x} + \frac{\partial\left[\frac{4}{3} * \mu * \frac{\partial u}{\partial x} - \frac{2}{3} * \mu\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]}{\partial x} + \frac{\partial\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]}{\partial y} + \frac{\partial\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]}{\partial z}$$

After cleaning up:

$$\rho\left(u * \frac{\partial u}{\partial x} + v * \frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Assume the effects of the wall to be confined to a region near the wall and call that region the boundary layer, δ_v ($\delta_v \ll x$).



$$\frac{\partial P}{\partial x} = 0$$

Scaling analysis

Given $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \frac{U_\infty}{x} \sim \frac{v}{\delta_v}$

Therefore

$$v \sim \frac{\delta_v}{x} * U_\infty$$

Given $\rho\left(u*\frac{\partial u}{\partial x}+v*\frac{\partial u}{\partial y}\right)=\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$

$$U_\infty*\frac{U_\infty}{x}, v*\frac{U_\infty}{\delta_v}\sim\nu(\frac{U_\infty}{x^2},\frac{U_\infty}{\delta_v^2})$$

$\frac{U_\infty}{x^2}\ll\frac{U_\infty}{\delta_v^2}$ since $\delta_v\ll x$ and can be neglected.

Therefore

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{U_\infty^2}{x}\sim\nu*\frac{U_\infty}{\delta_v^2}$$

$$\frac{\delta_v}{x}\sim\sqrt{\frac{\nu}{U_\infty x}}=\frac{1}{\sqrt{Re}}\text{(when } \delta_v\ll x, Re_x>100)$$

Energy equation

$$\rho C_v\left(\frac{\partial T}{\partial t}+u*\frac{\partial T}{\partial x}+v*\frac{\partial T}{\partial y}+w*\frac{\partial T}{\partial z}\right)+T\left(\frac{\partial P}{\partial T}\right)_v\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$
$$=\frac{\partial\left(k*\frac{\partial T}{\partial x}\right)}{\partial x}+\frac{\partial\left(k*\frac{\partial T}{\partial y}\right)}{\partial y}+\frac{\partial\left(k*\frac{\partial T}{\partial z}\right)}{\partial z}+\mu\emptyset_v+U'''$$

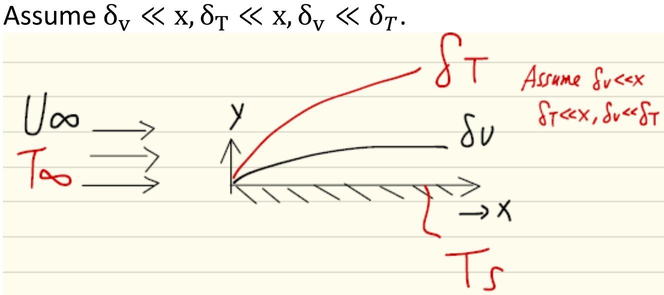
$\mu\emptyset_v$: viscous dissipation
U''': internal heat generation
Here we first neglect both viscous dissipation and internal heat generation

$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ as is given by the continuity equation.

Cleaning up and we get:

$$u*\frac{\partial T}{\partial x}+v*\frac{\partial T}{\partial y}=\frac{k}{\rho C_v}(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2})$$

Now consider a temperature boundary layer (δ_T).
Assume $\delta_v\ll x, \delta_T\ll x, \delta_v\ll\delta_T$.



Let $\Delta T=T_s-T_\infty$ and $\frac{k}{\rho C_v}=\frac{k}{\rho C}=\alpha$

Scaling analysis

$$\frac{U_\infty\Delta T}{x},\frac{\delta_v}{x}*\frac{U_\infty\Delta T}{\delta_T}\sim\alpha\left(\frac{\Delta T}{x^2},\frac{\Delta T}{\delta_T^2}\right)$$

Since $\delta_T\gg\delta_v$, $\delta_T\ll x$

Governing equation becomes:

$$u * \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}$$

$$\frac{U_\infty}{x} \sim \frac{\alpha}{\delta_T^2}$$

Plug in $\frac{\delta_v}{x} \sim \sqrt{\frac{\nu}{U_\infty x}}$

We get

$$\frac{\delta_v}{\delta_T} \sim \sqrt{\frac{\nu}{\alpha}} = \sqrt{\Pr}\ (\text{when Pr<0.01 } \delta_v \ll \delta_T)$$

$$q'' = -k * \frac{\partial T}{\partial y}|_{y=0} \rightarrow q'' \sim \frac{k\Delta T}{\delta_T}$$

$$h = \frac{q''}{\Delta T}$$

Therefore $h \sim \frac{k}{\delta_T}$

$$Nu_x = \frac{hx}{k} \sim \frac{x}{\delta_T} = \sqrt{Re_x Pr}$$

For $\delta_v \sim \delta_T$, $Nu_x \sim \sqrt{Re_x Pr}$

For $\delta_v > \delta_T$, $Nu_x \sim Re_x^{\frac{1}{2}} (\Pr)^{\frac{1}{3}}, \frac{\delta_v}{\delta_T} \sim \Pr^{\frac{1}{3}}$ for high Pr.

As a summary, for laminar boundary layer with $Re_x > 100$

$$\delta_v \sim \frac{x}{\sqrt{Re_x}}$$

$\Pr \rightarrow 0$, $Nu_x \sim \sqrt{Re_x Pr}$
$\Pr \rightarrow 1$, $Nu_x \sim \sqrt{Re_x Pr}$
$\Pr \rightarrow \infty$, $Nu_x \sim \sqrt{Re_x} Pr^{1/3}$
(neglect viscous dissipation)

Now let's consider viscous dissipation:

$$\emptyset_v = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2 - \frac{2}{3} * \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$

Since 2-D, incompressible, constant properties, steady flow, no body force, no U'''

$$\emptyset_v = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

$$\emptyset_v \sim \frac{U_\infty^2}{x^2}, \frac{\delta_v^2}{x^2} * \frac{U_\infty^2}{\delta_v^2}, \frac{U_\infty^2}{\delta_v^2} (dominant), \frac{\delta_v}{x} * \frac{U_\infty^2}{\delta_v x}, \frac{\delta_v^2}{x^2} * \frac{U_\infty^2}{x^2}$$

Since $\delta_v \ll x$

$$\emptyset_v \sim U_\infty^2 / \delta_v^2$$

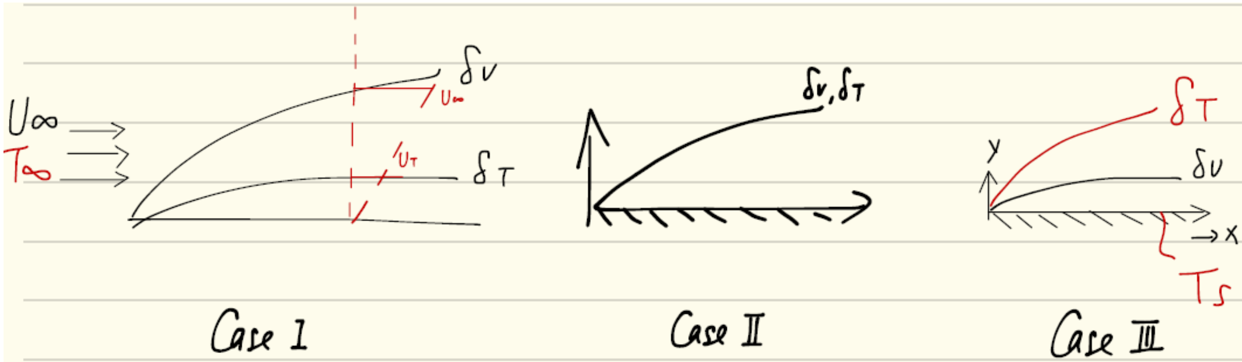
Therefore for a boundary layer with viscous dissipation

$$\text{Continuity equation: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{Momentum equation: } u * \frac{\partial u}{\partial x} + v * \frac{\partial u}{\partial y} = \nu * \frac{\partial^2 u}{\partial y^2}$$

$$\text{Energy equation: } u * \frac{\partial T}{\partial x} + v * \frac{\partial T}{\partial y} = \alpha * \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p} * \left(\frac{\partial u}{\partial y}\right)^2$$

Our results from before are unchanged for continuity and momentum. We consider three scenarios for energy.



Case one cannot happen. If δ_v is that big and dissipation matters, we can't have T_∞ at δ_T as kinetic energy is dissipated into heat.

Case two: $\delta_v \sim \delta_T = \delta$

$$U_\infty * \frac{\Delta T}{x}, \frac{\delta}{x} * \frac{U_\infty \Delta T}{\delta} \sim \frac{\alpha \Delta T}{\delta^2}, \frac{\nu}{C_p} * \frac{U_\infty^2}{\delta^2}$$

$$1 \sim \frac{k}{\rho C_p} * \frac{x}{U_\infty} * \frac{1}{\delta^2}, \frac{\nu}{C_p} * \frac{U_\infty^2}{\delta^2} * \frac{x}{U_\infty \Delta T}$$

$$1 \sim \frac{x^2}{\delta^2}\left[\frac{\mu}{\rho U_\infty x} * \frac{k}{C_p \mu}, \frac{U_\infty^2}{C_p \Delta T} * \frac{\nu}{U_\infty x}\right]$$

Recall $\Pr = \frac{C_p \mu}{k}, Re_x = \frac{U_\infty x}{\nu}$

And define $Ec = \frac{U_\infty^2}{C_p \Delta T}$ (Eckert number)

From previous analysis $\frac{x^2}{\delta^2} \sim Re_x$

$$1 \sim \frac{1}{\Pr}, Ec$$

$$\text{Advection} \sim \text{Diffusion, Dissipation}$$

For $\Pr \sim 1$,
if $Ec \ll 1$, advection~diffusion and we neglect dissipation;
if $Ec \sim 1$, advection~diffusion~dissipation;
if $Ec \gg 1$, not possible for $\delta_T \sim \delta_v$.

Case three: $\delta_T \gg \delta_v$

$$\frac{U_\infty \Delta T}{x}, \frac{\delta_v}{x} * \frac{U_\infty \Delta T}{\delta_T} \sim \frac{\alpha \Delta T}{\delta_T^2}, \frac{\nu}{C_p} * \frac{U_\infty^2}{\delta_v^2}$$

$$1 \sim \frac{k}{\rho C_p} * \frac{x}{U_\infty} * \frac{1}{{\delta_T}^2}, \frac{\nu}{C_p} * \frac{U_\infty^2}{{\delta_v}^2} * \frac{x}{U_\infty \Delta T}$$

$$1 \sim \frac{x^2}{\delta_v^2}\left[\frac{1}{Re_x Pr} * \frac{\delta_v^2}{\delta_T^2}, \frac{Ec}{Re_x}\right]$$

As before $\frac{x^2}{\delta_v^2} \sim Re_x$

$$1 \sim \frac{1}{\mathrm{Pr}} * \frac{\delta_v^2}{\delta_T^2}, Ec$$

For $\mathrm{Pr} \sim 1$ or $\mathrm{Pr} \gg 1$, advection~dissipation. $\mathrm{Ec} \sim 1$ $\quad u * \frac{\partial T}{\partial x} = \frac{\nu}{C_p} * \left(\frac{\partial u}{\partial y}\right)^2$

For $\sqrt{\mathrm{Pr}} \ll 1$ might get diffusion.

Friction and heat transfer

We have the model equations; they tell us about friction and heat transfer

$$\tau = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$q'' = h(T_s - T_\infty) = -k \left(\frac{\partial T}{\partial y}\right)_{y=0}$$

Only the gradient at the wall matters.

$$\bar{\tau} = \frac{1}{L}\int_0^L \tau dx$$

$$\bar{h} = \frac{1}{L}\int_0^L h dx$$

Note

$$\frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} = u * \frac{\partial u}{\partial x} + u * \frac{\partial u}{\partial x} + u * \frac{\partial v}{\partial y} + v * \frac{\partial u}{\partial y} = u * \frac{\partial u}{\partial x} + 0 + v * \frac{\partial u}{\partial y} = u * \frac{\partial u}{\partial x} + v * \frac{\partial u}{\partial y}$$

$$\frac{\partial (uT)}{\partial x} + \frac{\partial (vT)}{\partial y} = u * \frac{\partial T}{\partial x} + T * \frac{\partial u}{\partial x} + T * \frac{\partial v}{\partial y} + v * \frac{\partial T}{\partial y} = u * \frac{\partial T}{\partial x} + 0 + v * \frac{\partial T}{\partial y} = u * \frac{\partial T}{\partial x} + v * \frac{\partial T}{\partial y}$$

So we can write (assume no dissipation)

Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Momentum equation: $\frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} = \nu * \frac{\partial^2 u}{\partial y^2}$

Energy equation: $\frac{\partial (uT)}{\partial x} + \frac{\partial (vT)}{\partial y} = \alpha * \frac{\partial^2 T}{\partial y^2}$

Integrate continuity equation, use Leibnitz, go to $Y \geq \delta$:

$$\mathrm{d}\left(\int_0^Y u dy\right)/dx + v_Y - v_0 = 0$$

$$v_Y = -\frac{\mathrm{d}\left(\int_0^Y u dy\right)}{\mathrm{dx}}$$

Integrate momentum equation:

$$\frac{\mathrm{d}\left(\int_0^Y u^2 dy\right)}{\mathrm{dx}} + u_Y v_Y - u_0 v_0 = \nu\left[\left(\frac{\partial u}{\partial y}\right)_{y=Y} - \left(\frac{\partial u}{\partial y}\right)_{y=0}\right]$$

$$\frac{\mathrm{d}\left(\int_0^Y u^2 dy\right)}{\mathrm{dx}} + u_\infty v_Y = \nu\left[-\left(\frac{\partial u}{\partial y}\right)_{y=0}\right]$$

Integrate energy equation:

$$\frac{\mathrm{d}\left(\int_0^Y uT dy\right)}{\mathrm{dx}} + v_Y T_Y - v_0 T_0 = \alpha\left[\left(\frac{\partial T}{\partial y}\right)_{y=Y} - \left(\frac{\partial T}{\partial y}\right)_{y=0}\right]$$

$$\frac{\mathrm{d}\left(\int_0^Y uTdy\right)}{\mathrm{dx}} + v_Y T_\infty = \alpha[-\left(\frac{\partial T}{\partial y}\right)_{y=0}]$$

Substitute v_Y from continuity equation into the other two equations and rearrange.

$$\frac{\mathrm{d}\left(\int_0^{\delta_v} u(U_\infty - u)dy\right)}{\mathrm{dx}} = \nu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\frac{\mathrm{d}\left(\int_0^{\delta_T} u(T_\infty - T)dy\right)}{\mathrm{dx}} = \alpha\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

Alternatively,

$$\frac{\tau}{\rho} = \frac{\mathrm{d}\left(\int_0^{\delta_v} u(U_\infty - u)dy\right)}{\mathrm{dx}}$$

$$\frac{q''}{\rho C_p} = \frac{\mathrm{d}\left(\int_0^{\delta_T} u(T - T_\infty)dy\right)}{\mathrm{dx}}$$

To use these equatons

- Assume a velocity profile;
- Obtain 1^{st} order ordinary differential equation for $\delta(x)$;
- Solve for $\delta(x)$ and use profile to get τ.

There is no analytical solution available!!!!

Assume $\frac{u}{U_\infty} = f\left(\frac{y}{\delta}\right) = f(\eta)$

where $\eta = \frac{y}{\delta}$

$$f(\eta) = a + b\eta + c\eta^2 + d\eta^3$$

Boundary conditions:

$$u(x,0) = 0 \rightarrow a = 0$$

$$u(x,\delta) = U_\infty \rightarrow b + c + d = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{y=\delta} = 0 \rightarrow b + 2c + 3d = 0$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = 0 \rightarrow c = 0 \ \ \text{(at y=0, } u * \frac{\partial u}{\partial x} + v * \frac{\partial u}{\partial y} = \nu * \frac{\partial^2 u}{\partial y^2} = 0\text{)}$$

Let $\frac{u}{U_\infty} = \frac{3}{2} * \left(\frac{y}{\delta}\right) - \frac{1}{2} * \left(\frac{y}{\delta}\right)^3$

Solving O.D.E. of $\delta(x)$, we get $\delta(x) = 4.64\left(\frac{\nu x}{U_\infty}\right)^{0.5}$

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{Re_x}}$$

$$\tau_s = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{0.323\mu U_\infty Re_x^{0.5}}{x} = \frac{0.323\rho U_\infty^2}{\sqrt{Re_x}}$$

$$C_f = \frac{\tau_s}{\rho U^2/2}$$

$$C_f = \frac{0.646}{\sqrt{Re_x}}$$

Assume $\theta = \frac{T - T_s}{T_\infty - T_s} = e + f\eta + g\eta^2 + h\eta^3$
where $\eta = y/\delta_T$
Apply boundary conditions

$$\theta = \frac{3}{2} * \frac{y}{\delta_T} - \frac{1}{2} * \left(\frac{y}{\delta_T}\right)^3$$

Energy in integral form:

$$U_\infty \frac{d\left(\int_0^{\delta_T} \frac{u}{U_\infty} * (1-\theta)dy\right)}{dx} = \alpha \left(\frac{\partial \theta}{\partial y}\right)_{y=0}$$

Substitute the profiles of u and θ, for a case where $\delta_T \sim \delta_v \; or \; \delta_T \ll \delta_v \rightarrow Pr \sim 1 \; or \; Pr \gg 1$
Solve and we get

$$\frac{\delta_T}{x} = \frac{4.53}{Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}}$$

$Nu_x = \frac{hx}{k} = \frac{q''x}{(T_s - T_\infty)k} = -x * \frac{\partial \theta}{\partial y}_{y=0} = \frac{3}{2} * \frac{x}{\delta_T} = 0.331 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$ (in accord with the results from scaling analysis)

Similarity solutions to boundary layer equations

(Through similarity (introduction of φ), we turn PDE to ODE so that we can solve.)

Steady state, constant properties, 2-d, no body force in x, neglect ∅, no U''', boundary layers, no $\frac{\partial P}{\partial x}$

$$\text{Continuity equation: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{Momentum equation: } u * \frac{\partial u}{\partial x} + v * \frac{\partial u}{\partial y} = \nu * \frac{\partial^2 u}{\partial y^2}$$

$$\text{Energy equation: } u * \frac{\partial T}{\partial x} + v * \frac{\partial T}{\partial y} = \alpha * \frac{\partial^2 T}{\partial y^2}$$

Define

$$u = \frac{\partial \varphi}{\partial y}, -v = \frac{\partial \varphi}{\partial x}$$

Continuity is satisfied, substitute this into momentum equation.

$$\frac{\partial \varphi}{\partial y} \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial y^2} = \nu * \frac{\partial^3 \varphi}{\partial y^3}$$

Out of the blue:
Assume $\varphi = H(\varepsilon) f(\eta)$

$$\varepsilon = x$$

$$\eta = y\sqrt{\frac{U}{\nu x}}, H = \sqrt{\nu U \varepsilon}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial \varepsilon} * \frac{\partial \varepsilon}{\partial y} + \frac{\partial \varphi}{\partial \eta} * \frac{\partial \eta}{\partial y} = 0 + \sqrt{\nu U \varepsilon} * f' * \sqrt{\frac{U}{\nu x}} = Uf' = u$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \varepsilon} * \frac{\partial \varepsilon}{\partial x} + \frac{\partial \varphi}{\partial \eta} * \frac{\partial \eta}{\partial x} = \sqrt{\frac{U\nu}{x}} * \frac{1}{2} * (f - \eta f') = -v$$

$$\frac{\partial^2 \varphi}{\partial y^2} = U\sqrt{\frac{U}{\nu x}} * f''$$

$$\frac{\partial^3\varphi}{\partial y^3}=\frac{U^2}{\nu x}*f'''$$
$$\frac{\partial^2\varphi}{\partial x\,\partial y}=-\frac{1}{2x}U\eta f''$$

Substitute into momentum and rearrange

$$f'''+\frac{1}{2}*f*f''=0 \text{ (magically…)}$$

where $f'=df/d\eta$

Boundary conditions:

$u(x,0)=0$ when y=0 $\eta=0$ for x>0

$$Uf'_{\eta=0}=0$$

Therefore @ $\eta=0$ $f'=0$

$v(x,0)=0$

Therefore @$\eta=0$ $f=0$

$u(x,\infty)=U$

Therefore $f'_{\eta\to\infty}=1$

The nonlinear PDE modeling momentum is now ODE. Solve numerically.

How do we determine what to assume?

The boundary layer equations model an artificial thing. No natural length scale. We contrived δ.

In the boundary layers, $\frac{y}{\delta}\to$ goes from 0 to 1.

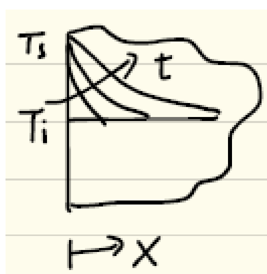
Scale analysis showed

$\frac{\delta}{x}\sim\sqrt{\frac{\nu}{Ux}}$

Therefore $\frac{y}{\delta}\sim y\sqrt{\frac{U}{\nu x}}\to\eta$

This is how we get our assumption! Group variables together and turn PDE to ODE!!!

Recall our discussion



$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}*\frac{\partial T}{\partial t}$

$T(x,0)=T_i$

$T(x\to\infty,t)=T_i$

$T(0,t>0)=T_s$

δ is the penetration depth.

$\Delta T=T_s-T_i$

Scale:

$$\frac{\Delta T}{\delta^2}\sim\frac{1}{\alpha}*\frac{\Delta T}{t}$$
$$\delta\sim\sqrt{\alpha t}$$
$$\frac{x}{\delta}\sim\frac{x}{\sqrt{\alpha t}}$$

Let $\eta=\frac{x}{\sqrt{\alpha t}}$ and $T(x,t)=f(\eta)$ we have $f''+\frac{1}{2}*f'\eta=0$

Similarity steps:

1. Group variables together;
2. Turn PDE to ODE.

The thermal boundary layer

$$u*\frac{\partial T}{\partial x}+v*\frac{\partial T}{\partial y}=\alpha*\frac{\partial^2 T}{\partial y^2}$$

$T(x,0)=T_s$

$T(0,y)=T_\infty$

$T(x,\infty)=T_\infty$

Take $\theta=\frac{T-T_s}{T_\infty-T_s}$

$$u*\frac{\partial \theta}{\partial x}+v*\frac{\partial \theta}{\partial y}=\nu/Pr*\frac{\partial^2 \theta}{\partial y^2}$$

$\theta(x,0)=0, \theta(0,y)=1, \theta(x,\infty)=1$

Define $\theta_o(\eta)=\theta(x,y), \eta=y\sqrt{\frac{U}{\nu x}}$

The transform will yield $\theta_o''+\frac{Pr}{2}*f\theta_o'=0, \theta_o(0)=0, \theta_o(\infty)=1$

$Nu_x=\theta_o'\sqrt{Re_x}$ → Solve numerically!

The numerical solution is very closely approximated with

$\frac{Nu_x}{\sqrt{Re_x}}=0.564Pr^{1/2}$ Pr<0.05

$\frac{Nu_x}{\sqrt{Re_x}}=0.33206Pr^{1/3}$ 0.6<Pr<10

$\frac{Nu_x}{\sqrt{Re_x}}=0.339Pr^{1/3}$ Pr>10